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FOREWORD

This report was prepared by Bolt Beranek and Newman Inc., Cambridge, Massachusetts, for the Aerospace Dynamics Branch, Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio 45433, under Contract No. AF 33(615)-2649. The research performed is part of a continuing effort to provide advanced techniques in the application of random process theory and statistics to vibration problems. This effort is part of the Research and Technology Division, Air Force Systems Command's exploratory development program. The contract was initiated under Project No. 1370, "Dynamic Problems in Flight Vehicles," Task No. 137005, "Prediction and Control of Structural Vibration." Mr. R. G. Merkle of the Vehicle Dynamics Division, FDDS was the project engineer.

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This report has been reviewed and is approved.

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ABSTRACT

The concepts and relations which form the theoretical foundation of the statistical energy approach to vibration analysis are delineated. The utility of this approach for dealing with complex systems is discussed, and its range of applicability is indicated.

The important properties of the modes of vibrating systems are reviewed and used to exhibit relations between modal responses and average responses of total systems. It is demonstrated that under some conditions, which are often approximated in actual systems, the average rate of flow of energy from one mode to another is proportional to the difference in the modal energies. It is also shown that under some conditions the average rate of flow of energy between two sets of modes (representing groups of modes of two coupled systems in a given frequency band) is proportional to the difference in the set-average modal energies. Examples are presented which indicate how these relations permit one to obtain approximate solutions to complex problems simply, on the basis of energy conservation considerations.

Available extensions of the major concepts developed in detail are mentioned, and references to the current literature are given.

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INTRODUCTION

Recent years have seen a continuation and acceleration of the trend toward higher performance vehicles, toward increased propulsion system power, and toward more highly sophisticated equipment and instrumentation. This trend has brought with it a greater incidence of problems associated with high frequency vibrations and has caused aerospace engineers to concern themselves increasingly with vibrations at frequencies considerably above the fundamental structural resonances.

Classically, vibration engineers have focused their attention on low-frequency oscillations, since the lowest few vibration modes are generally the ones which are associated with the greatest deflections, the highest stresses, and gross structural failures. However, the rather complete arsenal of analysis techniques which have been developed for dealing with low-frequency vibration problems contains none which can deal simply and effectively with most high-frequency problems, such as those of importance in relation to sonically induced fatigue, instrumentation performance, or noise transmission.

Although the classical methods are valid in principle at all frequencies, their use is in fact very often impractical for high frequencies, particularly for randomly excited complex structures. The classical approach consists of determining the natural modes, of calculating the responses of these modes to a specified excitation of interest, and of superposing these responses to determine the total structural response. Continuous structures have an infinite number of modes, but generally only the lowest few of these are of importance in low frequency vibrations, so that in these cases one needs to consider only those few modes. At high frequencies, however, a frequency band of interest usually encompasses the resonances of a large number of modes, and one must consider the responses of all of these modes in calculating the structural vibrations in that band.

Such classical multimodal analyses may require an amount of computation which is so great that it exceeds the capacities of some of today's largest computers. But, one also encounters another, perhaps more significant, limitation on the utility of such analyses. Since the shapes of the higher modes are much more sensitive to details of the structure than are those of the lower modes, one must be able to describe the structural and material properties

with greater precision in order to be able to perform higher mode calculations meaningfully. The required precision can generally be achieved only for fictitious structures, in view of the manufacturing tolerances and the variation in material properties one inevitably encounters in realistic cases.

If one could perform a high-frequency multimodal analysis meaningfully, one would still be faced with the problem of interpreting the large mass of results that is generated in such a set of calculations. These results would be descriptions of the motions, within the frequency band of interest, of a large number (ideally all) of the points of the structure. In general, these motions will be complex and different for different points, and one will need to perform additional calculations, e.g. to determine average values of various response properties, in order to reduce the large amount of information to tractable size.

The statistical energy analysis approach to structural vibrations problems, pioneered by R. H. Lyon, P. W. Smith, Jr., and I. Dyer, was developed in response to the need for a simple means for understanding and estimating significant properties of multimodal vibrations of complex systems. This approach was spawned by the realization that averaging initially, and then carrying out calculations in terms of average quantities, should lead to results much more readily than the classical approach, which involves much initial detailed calculation and subsequent averaging.

The statistical energy approach to some extent is analogous to the "room acoustics" approach for dealing with sound in architectural spaces. In the latter approach one does not attempt to solve the acoustic wave equation in detail for the complex spaces involved; rather, one analyzes the average behavior of energy variables and interprets the results in terms of averages of dynamic variables (e.g., pressures, velocities). In the statistical energy approach one similarly does not concern oneself with a detailed solution of the system equations of motion, but one analyzes average energy distributions and relates those to average response variables.

Although the term "statistical energy method" has come into somewhat popular use at this writing, there really exists no "method" consisting of a well-defined technique leading to a guaranteed result (like the Rayleigh-Ritz or the Holzer methods, for example). There does exist an

approach, which - like room acoustics - is one of a set of tools, the intelligent use of which can give one approximate, but simple and useful answers to problems which could not otherwise be solved realistically.

The fundamentals of the statistical energy analysis approach are scattered in the technical literature, generally are couched in language which may discourage the non-specialist, and in some cases have been only implied, and not spelled out in detail. Consequently, this potentially highly useful approach has not come to be used as widely as one may have expected on the basis of its simplicity of application. It is the purpose of this report to fill the apparent need for a presentation of the fundamentals of the statistical energy analysis approach in easily accessible form, for an explanation of its area of applicability, and for an illustration of how these fundamentals may be used.

The first of the following sections reviews the concepts and properties of modes of structural vibrations, and points out some useful relations between modal response and total average response properties. The second section derives the basic relations that govern the exchange of energy between two coupled modes. The third section generalizes this relation to enable one to determine the flow of energy from one set of modes, representing one structure or fluid system, to another set of modes, representing another such system. The fourth section relates some of the parameters introduced in the modal analyses to more commonly employed, and more easily estimated and measured system parameters. The fifth and final section illustrates the application of the statistical energy approach to a special class of problems.

The reader who does not have the inclination to wade through the considerable amount of mathematical manipulation which appears in this report should not be discouraged. He should ignore all those details and concentrate on the results and conclusions indicated in the "Summary and Conclusions" subsections. The strong point of the statistical energy approach, after all, is the simplicity with which the central results can be stated and applied.

MODES OF CONTINUOUS SYSTEMS

If an undamped linear elastic system vibrates in the absence of external forces in such a way that all points in the system move sinusoidally in time at the same frequency, then the system is said to vibrate in a "natural mode" - or "mode", for short. This type of vibration is a very special member of the class of all possible types of free vibrations, and possesses some useful properties, several of which are reviewed in the present section.

The various points in an undamped system vibrating in a natural mode generally oscillate at different amplitudes, but are either completely in phase or out of phase with each other. All points reach their maximum excursion (some in the negative and some in the positive direction) at the same instant, all points pass through their equilibrium positions simultaneously, and all points reach any fraction of their maximum excursions simultaneously. Thus, a natural mode oscillation may be characterized by a spatial distribution of deformations (e.g., by the distribution of deflections at any suitable instant), multiplied by a sinusoid in time and a constant which is indicative of the amplitudes.

The spatial distribution of deformations is called a "mode shape". Functions describing a given mode shape may differ from each other by a multiplicative constant, since the definition of the mode shape is arbitrary to that degree. However, once a mode shape function has been selected, the constant by which one must multiply this function so that it represents a given modal vibration is fully determined. This latter constant is called "modal amplitude"; a specified modal amplitude and mode shape function together define a unique distribution of deformations.

The frequency at which a system in absence of external forces vibrates with a given mode shape is called the "natural frequency" corresponding to that mode. A continuous elastic system can vibrate in an infinite number of different modes, with an infinite number of corresponding natural frequencies.

The discussion of modes presented here is of necessity limited in scope and rigor. A clearly written more complete treatment may be found in Reference 1; a collection of information on the modal properties of simple systems is available in Reference 2.

Mode Shapes and Natural Frequencies

The linear partial differential equations that govern the motions of undamped continuous elastic systems (such as strings, membranes, shafts, beams and plates) in absence of external forces may be written in the form

$$\mu \ddot{u} + \tilde{K} u = 0 \quad . \tag{1}$$

Here $\tilde{K}$ is a linear differential operator which involves only the spatial coordinates and represents the stiffness characteristics of the system, and μ is a function of only the coordinates and describes the mass distribution of the system. The symbol u denotes the deflection of the system from equilibrium as a function of time t and of the spatial coordinates, and the dot atop a symbol indicates differentiation with respect to time.

[For example, for a shaft in torsion u=u(x,t) denotes the torsional angular deflection as a function of the position along the shaft and of time. The corresponding operators for Eq. (1) are

$$\mu = \rho J, \ \tilde{K} = -\frac{\partial}{\partial x}\left(K_s G \frac{\partial}{\partial x}\right) \quad ,$$

where ρ denotes the material density, J the polar moment of inertia of the cross-sectional area, K_s the torsional constant of the crosssection, G the shear modulus of the material. In the most general case all of these parameters may be functions of the coordinate x measured along the length of the shaft.

Similarly, for a flat plate in flexure, u=u(x,y,t) denotes the displacement of the plate normal to its midsurface, as a function of coordinates x,y along that midsurface and of time. Here

$$\mu = \rho_s(x,y), \ \tilde{K} = \nabla^2(D \nabla^2) \quad ,$$

where ρ_s denotes plate mass per unit surface area, and D represents the plate flexural rigidity, usually given as $Eh^3/12(1-\nu^2)$ for homogeneous plates of thickness h, Young's modulus E and Poisson's ratio ν. The symbol ∇^2 represents the "Laplacian" operator; in the usual Cartesian coordinates $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In the most general isotropic case, ρ_s and D may both vary with x and y.]

Natural modal vibrations are defined as free (unforced) vibrations in which all points move in unison, sinusoidally in time, at the same frequency. Such vibrations correspond to solutions of Eq. (1) which are of the form[1]

$$u(x,y,t) = \psi(x,y)\ [B\cos(\omega t + C)] \quad , \tag{2}$$

in which the spatial and time dependences are separated and where B and C are constants. Substitution of Eq. (2) into (1) yields the time-independent partial differential equation

$$\tilde{K}\psi - \omega^2\mu\psi = 0 \tag{3}$$

which the function $\psi(x,y)$ of the spatial coordinates must satisfy.

One finds that solutions ψ of Eq. (3), which satisfy the boundary (edge) conditions applicable in a given case and which are not identically zero, exist only for certain values of the frequency ω. These values are called the natural frequencies, and the corresponding solution functions $\psi(x,y)$ are called the mode shapes of the system. To each n-dimensional system (n=1,2,3) there correspond n-fold infinite sets of natural frequencies and mode shapes. To each two-dimensional system (plate, membrane) there correspond doubly infinite sets of natural frequencies ω_{jk} and mode shapes $\psi_{jk}(x,y)$; $j,k = 1,2,\cdots$, which satisfy Eq. (3)[1] or

$$\tilde{K}\psi_{jk}(x,y) - \omega_{jk}^2\ \mu\psi_{jk}(x,y) = 0 \quad . \tag{4}$$

Orthogonality and Normalization of Mode Shapes

The mode shapes of systems whose edges are elastically constrained against lateral and rotational deformation[2] possess

[1]From here on the notation used is that for two-dimensional systems. Expressions applicable to one- and three-dimensional systems can be deduced by direct analogy. (Also, see Ref. 2.)

[2]Zero and infinite values of constraint stiffness are permitted, but in the general case the boundaries exert restoring forces which are proportional to the deformations and/or restoring moments which are proportional to the rotations. Infinite lateral stiffness and zero rotational boundary stiffness corresponds to the usual "simple support" conditions. The case where both stiffnesses are infinite corresponds to clamped edges; that where both are zero corresponds to free edges. (p. 269, Ref. 1.)

the useful property of "orthogonality". Mathematically, this means that the mode shapes satisfy

$$\iint_{A_s} \mu(x,y)\ \psi_{jk}(x,y)\ \psi_{j'k'}(x,y)\ dA_s = \begin{cases} m\Psi_{jk} & \text{for } j=j',\ k=k' \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where the integration is carried out over the entire elastic system (here shown as over the entire area A_s of a two-dimensional system). In other words, if the integral of Eq. (5) involves two different modes, then the result is zero, but if it involves the same mode twice, then the result is a finite number. This number has been denoted by $(m\Psi_{jk})$, where m denotes the total mass of the system, and satisfies

$$m\Psi_{jk} \equiv \iint_{A_s} \mu(x,y)\ \psi_{jk}^2(x,y)\ dx\ dy \quad , \tag{6}$$

in accordance with Eq. (5).

One may observe that multiplication of ψ by an arbitrary constant does not affect Eq. (3) and that hence this equation defines $\psi(x,y)$ only within an arbitrary multiplicative constant. Equation (6) therefore involves a similar arbitrariness. It is generally useful for analysis purposes to "normalize" the mode shapes $\psi_{jk}(x,y)$, - that is, to set down some more or less arbitrary rule for determining the multiplicative constant to be used in a given analysis. One commonly used rule (but by no means the only one) requires $\Psi_{jk}=1$ for all j and k, so that $m\Psi_{jk}$ in Eq. (6) reduces to the total system mass m. If this rule is used, then the mode shape functions satisfy

$$\iint_{A_s} \mu(x,y)\ \psi_{kj}^2(x,y)\ dx\ dy = m = \iint_{A_s} \mu(x,y)\ dx\ dy \tag{7}$$

in addition to Eq. (4).

[For example, for torsional vibrations of a uniform shaft clamped at one end (x=0) and free at the other (x=L), one finds

$$\psi_j(x) = \sqrt{2}\,\sin[(2j-1)\pi x/2L], \quad \omega_j = [(2j-1)\pi/2L]\sqrt{K_s G/\rho J}.$$

For a uniform rectangular plate extending from x=0 to x=a and from y=0 to y=b, and simply supported on all four edges one finds

$$\psi_{jk}(x,y) = 2\,\sin(j\pi x/a)\,\sin(k\pi y/b) \quad ,$$

$$\omega_{jk} = \pi^2(j^2/a^2 + k^2/b^2)\,\sqrt{D/\rho_s h} \quad .]$$

Representation of Deformations in Terms of Modes

Any mathematically "well-behaved" system deflection $u(x,y,t)$ may be expressed as a sum

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} U_{jk}(t)\,\psi_{jk}(x,y) \qquad (8)$$

involving the mode shapes ψ_{jk} and "modal amplitude coefficients" $U_{jk}(t)$, which depend only on time (Refs. 1,3). To evaluate these coefficients one may proceed in a manner which is analogous to that used to obtain a Fourier series expansion of a given function. If one multiplies both sides of Eq. (8) by $\mu(x,y)\psi_{j'k'}(x,y)$ and integrates over the entire elastic system, then one finds in view of the orthogonality relation (5) that only one of the integrals of the sum will not vanish. (Namely, that for which j=j' and k=k'.) With the normalization of Eq. (7) the value of this non-zero integral is m, and thus one obtains the result

$$U_{jk}(t) = \frac{1}{m}\iint_{A_s} u(x,y,t)\,\psi_{jk}(x,y)\,\mu(x,y)\,dx\,dy \qquad (9)$$

which permits one to determine the coefficient $U_{jk}(t)$ which corresponds to the mode shape $\psi_{jk}(x,y)$ for a given deformation $u(x,y,t)$.

Modal Analysis of System Motions

The deflection of a viscously damped elastic system due to the action of a distributed load $p(x,y,t)$ is governed by a differential equation of the form

$$\mu \ddot{u} + \gamma \dot{u} + \tilde{K} u = p(x,y,t) \tag{10}$$

where the various terms have the same meaning as in Eq. (1), and γ is an operator or function which involves only the coordinates and which accounts for energy dissipation (viscous damping) in the system.

In order to determine the equations that govern the responses of the modes of the system one may substitute for $u(x,y,t)$ in Eq. (10) the series given in Eq. (8). If one uses Eq. (4) to replace the terms involving $\tilde{K}$ by others involving μ, one obtains

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu \ddot{U}_{jk} + \gamma \dot{U}_{jk} + \mu \omega_{jk}^2 U_{jk} \right) \psi_{jk}(x,y) = p(x,y,t). \tag{11}$$

Now one may again try to use the strategem that was employed in the derivation of Eq. (9). Namely, one may multiply both sides of Eq. (11) by $\psi_{j_1k_1}(x,y)$, integrate over the entire system, and apply the orthogonality condition. However, in general the damping operator γ and the mass distribution μ may not be related in any manner which permits one to apply to the γ terms the orthogonality condition of Eq. (5) which involves μ.

Fortunately, the special case where $\gamma = \beta\mu$, with β a constant, turns out to be a useful one, and may serve as a good approximation for many kinds of lightly damped systems. In this case the previously discussed strategem does work. Its use in conjunction with the normalization of Eq. (7) yields a result which one may write as

$$\boxed{m \ddot{U}_{jk} + c \dot{U}_{jk} + K_{jk} U_{jk} = F_{jk}(t)} \quad , \tag{12}$$

if one defines

$$c \equiv m\,\beta \qquad K_{jk} \equiv m\,\omega_{jk}^2$$

$$F_{jk}(t) \equiv \iint_{A_s} p(x,y,t)\,\psi_{jk}(x,y)\,dx\,dy\,. \tag{13}$$

For a given elastic system subject to a given loading, the "modal mass" m, the "modal damping coefficient" c, and the "modal stiffness" K_{jk} are constant, whereas the "modal force" F_{jk} is a function of time only. Equation (12), which governs the modal coefficients U_{jk}, may be recognized to be the same as the well-known equation that governs the displacement of a linear single-degree-of-freedom system consisting of a mass m, mounted on parallel arrangement of a spring (with stiffness K_{jk}) and dashpot (with viscous damping coefficient c), and subject to a force $F_{jk}(t)$ which varies with time.

It is of interest to note that Eq. (12) applies for each mode (i.e. for all values of j and k), but that the equation for a given j and k does not involve terms in other values of j and k. There is then no interaction between the modal motions, and the modes are said to be "uncoupled". Such uncoupling does not occur in the most general case, in which orthogonality cannot be applied to the damping terms involving γ, and in which a summation involving all modes would appear in Eq. (12) instead of the single $c\,U_{jk}$ term. However, as mentioned previously, one may usually ignore this damping coupling for lightly damped systems.

Spatial Averages of Deformations; System Kinetic Energy

If one squares both sides of Eq. (8), multiplies the result by μ, and integrates over the entire system, one obtains

$$\iint_{A_s} u^2(x,y,t)\,\mu(x,y)\,dx\,dy = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j'=1}^{\infty}\sum_{k'=1}^{\infty}\iint_{A_s} U_{jk}\,U_{j'k'}\psi_{jk}\psi_{j'k'}\mu\,dx\,dy$$

$$= m\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} U_{jk}^2(t) \tag{14}$$

where the second form of the right-hand side follows from orthogonality, Eq. (5), and from Eq. (7). The weighted spatial average value of u^2, with the averaging done over the entire system and with μ being used as a weighting function, is defined as

$$\overline{u^2(t)} = \left[\iint_{A_s} u^2(x,y,t)\ \mu(x,y)\ dx\ dy\right]\cdot\left[\iint_{A_s} \mu(x,y)\ dx\ dy\right]^{-1} . \tag{15}$$

By comparing this expression with Eq. (14) and (7) one finds that

$$\boxed{\overline{u^2(t)} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} U_{jk}^2(t)} \tag{16}$$

and thus that the weighted mean square displacement is equal to the sum of the squares of all the modal displacements $U_{jk}(t)$. For uniform systems, in which the mass distribution μ is constant, the weighted mean-square displacement is equal to the ordinary mean-square displacement, as may easily be verified from Eq. (15); Eq. (16) then applies to the ordinary mean-square displacement.

If one writes an expression like Eq. (14) in terms of the velocity $v(x,y,t)$ and the corresponding modal velocity coefficients V_{jk}, then the integral on the left-hand side of that expression may be seen to represent twice the total kinetic energy T_T of the system. Thus, one may write

$$T_T(t) = \frac{1}{2}\iint_{A_s} v^2(x,y,t)\ \mu\ dx\ dy = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{jk}(t) \tag{17}$$

where

$$T_{jk}(t) = \frac{m}{2} V_{jk}^2(t) \tag{18}$$

denotes the kinetic energy associated with mode (j,k). Equation (17) shows that the total kinetic energy is equal to the sum of all of the modal kinetic energies, and also to $\frac{1}{2} m \overline{v^2}$, where the weighted mean-square velocity $\overline{v^2}$ is defined analogously to $\overline{u^2}$ in Eq. (15).

Expressions for the time- and space- averages may be obtained, of course, simply by averaging both sides of Eqs. (16)-(18) with respect to time.

Summary and Conclusions

The definition of modes of elastic systems and the properties of such modes have been reviewed in rather general terms in the present section.

It has been indicated that any physically reasonable displacement, velocity, etc., distribution can be represented as a superposition of corresponding modal distributions.

Orthogonality of the modes, which applies for systems with elastically constrained boundaries (and thus for all the usually considered ideal cases and for most of those generally of practical interest), has been shown to lead to uncoupled equations of motions for the individual modes, except where interaction occurs due to damping.

The uncoupled modal equations of motion have been demonstrated to be precisely of the same form as those for independent single-degree-of-freedom mass-spring-dashpot systems. Thus, by use of the concept of modes one may replace the complex problem of determining the motions of a continuous distributed elastic system by the much simpler problem of evaluating the motions of single-degree-of-freedom systems which represent the modes of the elastic system. The motions of the total system may then be obtained by superposition of all of the modal motions, spatial averages of motion quantities may easily be determined from the corresponding modal amplitude coefficients, and the total kinetic energy of the system may be found simply by adding all of the modal kinetic energies.

POWER FLOW BETWEEN TWO MODES

In many practical problems involving the vibrations of complex elastic systems, particularly where the excitations and responses are random, one is more concerned with understanding the interactions of various parts of complex elastic systems and with estimating average vibration levels, than with obtaining more detailed information about the system vibrations. One finds that energy analyses are particularly useful in such cases. These analyses tend to be simple, since they can make use of the easily applied concept of conservation of energy. And, as evident from the previous section, if one knows the kinetic energy of a system one knows its mean-square velocity.

Before concerning oneself with the interaction of two (or more) interconnected elastic systems, where each system contains many modes, one may do well to develop an understanding of the interaction of just two modes. It is clear that in the steady state the vibrational energy of a mode of a two-mode system is affected by the power supplied to the mode from external sources, by the power dissipated by the mode, and also by the power that the mode in question receives from (or supplies to) the other mode. The directly supplied and the dissipated powers can generally be evaluated and related to modal parameters and energies with relative ease. In order to make an energy approach useful, however, one also needs expressions which relate the mode-to-mode power flow to modal energies. It is the purpose of the present section to develop such expressions.

It has been pointed out that a mode of an elastic system behaves precisely like an ideal single-degree-of-freedom system consisting of the modal mass mounted on a spring (having the modal stiffness) and a dashpot (with the modal damping coefficient). Consequently, the energy transfer between two modes is studied here by analyzing the corresponding behavior of two coupled mass-spring-dashpot systems, as shown schematically in Fig. 1. In order to facilitate the analysis, and since linear systems are of primary interest, all of the present discussion is restricted to linear coupling.

The concepts of impedance and admittance are reviewed in the first of the following sections. Thereafter the restrictions which the assumption of conservative coupling places on these quantities are delineated. (Conservative coupling is coupling which neither dissipates nor supplies energy, and thus is of

primary interest where one wishes to study the energy flow between systems.) Next are developed expressions for the modal kinetic energies. Finally, it is demonstrated that the time-average power flow between two modes is proportional to the difference in the time-average modal kinetic energies under some conditions which are often approximated in practice.

Admittances and Impedances

In order to maintain as much generality as possible, it is useful initially to place no restriction on the coupling element L of Fig. 1, except for requiring it to be linear. This restriction implies that L can give rise only to linear terms in the equations of motion, so that response variables (displacements, velocities, etc.) are proportional to excitation variables (forces, rate of change of force, etc.), regardless of any of the amplitudes involved.

The properties of linear systems subject to purely sinusoidal excitations may be studied conveniently by use of phasor notation (Ref. 4). In this notation each sinusoidally varying dynamic variable is represented by a complex quantity or "phasor", and the actual time functions are obtained by taking the real part of the product of the corresponding phasor and $e^{-i\omega t}$, where ω denotes the radian frequency, t time, and $i = \sqrt{-1}$. Thus, for example, if $\underline{V}_j$ is the phasor corresponding to the velocity $V_j(t)$, then

$$V_j(t) = \mathrm{Re}[\underline{V}_j\, e^{-i\omega t}] = \mathrm{Re}[\underline{V}_j]\cdot\cos\omega t + \mathrm{Im}[\underline{V}_j]\cdot\sin\omega t \quad . \qquad (19)$$

In view of the linearity of the two-mass system of Fig. 1 the velocity phasors of the two masses obey the relation

$$\begin{aligned} \underline{V}_1 &= Y_{11}\,\underline{F}_1 + Y_{12}\,\underline{F}_2 \\ \underline{V}_2 &= Y_{21}\,\underline{F}_1 + Y_{22}\,\underline{F}_2 \end{aligned} \qquad (20)$$

where $\underline{F}_1$ and $\underline{F}_2$ denote phasors corresponding to the forces F_1 and F_2. The quantities Y_{rs} are called "admittances"; from Eqs. (20) it is evident that $Y_{11} = \underline{V}_1/\underline{F}_1$ if $\underline{F}_2 = 0$, $Y_{12} = \underline{V}_1/\underline{F}_2$ if $\underline{F}_1 = 0$, $Y_{21} = \underline{V}_2/\underline{F}_1$ if $\underline{F}_2 = 0$, etc.

By solving Eqs. (20) for $\underline{F}_1$ and $\underline{F}_2$ one finds that one may write

$$\underline{F}_1 = Z_{11}\,\underline{V}_1 + Z_{12}\,\underline{V}_2$$
$$\underline{F}_2 = Z_{21}\,\underline{V}_1 + Z_{22}\,\underline{V}_2 \qquad (21)$$

where

$$Z_{11}=Y_{22}/D, \quad Z_{22}=Y_{11}/D, \quad Z_{12}=-Y_{12}/D, \quad Z_{21}=-Y_{21}/D$$
$$D\equiv Y_{11}Y_{22} - Y_{12}Y_{21} \quad . \qquad (22)$$

The Z_{rs} quantities are called "impedances". From Eqs. (21) one may observe that $Z_{11}=\underline{F}_1/\underline{V}_1$ if $\underline{V}_2=0$, $Z_{12}=\underline{F}_1/\underline{V}_2$ if $\underline{V}_1=0$, etc. Alternately, by solving Eqs. (21) for V_1 and V_2 one obtains Eqs. (20) and finds that the admittances may be expressed in terms of the impedances as

$$Y_{11}=Z_{22}/E, \quad Y_{22}=Z_{11}/E, \quad Y_{12}=-Z_{12}/E, \quad Y_{21}=-Z_{21}/E$$
$$E\equiv Z_{11}Z_{22}-Z_{12}Z_{21}=1/D \quad . \qquad (23)$$

The impedances and admittances in general are complex quantities and functions of the frequency ω. They depend in general on the uncoupled system parameters, as well as on the coupling element.

Implications of Conservative Coupling

In order to study the flow of energy between systems it is useful to introduce the restriction of conservative coupling in addition to that of linear coupling; that is, to postulate coupling which neither dissipates nor supplies energy. Only for such coupling is the energy leaving one of two coupled systems equal to the energy reaching the other, and only for such coupling can one speak unambiguously of energy flow from one system to the other. The present section consequently is concerned with the restrictions which are imposed on the various admittances and impedances by the requirement that the coupling be conservative.

Conservation of energy demands that in the steady state the total work done per cycle by the external forces be just equal to the total energy dissipated per cycle. In view of the well-known relations between power and forces and velocities in complex notation, as also summarized in Ref. 5, this statement of energy conservation may be written as

$$\tfrac{1}{2}\,\mathrm{Re}[\underline{F}_1\underline{V}_1^*] + \tfrac{1}{2}\,\mathrm{Re}[\underline{F}_2\underline{V}_2^*] = \tfrac{1}{2}|\underline{V}_1|^2\,\mathrm{Re}[Z_{11}] + \tfrac{1}{2}|\underline{V}_2|^2\,\mathrm{Re}[Z_{22}]\ , \tag{24}$$

where the asterisk indicates the complex conjugate of the quantity to which it is appended.

Substitution for F_1 and F_2 from Eqs. (21), replacement of $|\underline{V}_1|^2$ by $\underline{V}_1\underline{V}_1^*$, and elimination of terms common to both sides of the resulting equation, permits one to reduce the foregoing relation to

$$\mathrm{Re}[Z_{12}\,\underline{V}_2\,\underline{V}_1^* + Z_{21}\,\underline{V}_1\,\underline{V}_2^*] = 0\ . \tag{25}$$

Since $\underline{V}_2\underline{V}_1^* = (\underline{V}_1\underline{V}_2^*)^*$, this result implies

$$\begin{aligned}(\mathrm{Re}[Z_{21}] + \mathrm{Re}[Z_{12}])\,\mathrm{Re}(\underline{V}_1\underline{V}_2^*) - \\ (\mathrm{Im}[Z_{21}] - \mathrm{Im}[Z_{12}])\,\mathrm{Im}(\underline{V}_1\underline{V}_2^*) = 0.\end{aligned} \tag{26}$$

But this relation must hold for all $\underline{V}_1$ and $\underline{V}_2$, and thus for all values of $\mathrm{Re}(\underline{V}_1\underline{V}_2^*)$ and $\mathrm{Im}(\underline{V}_1\underline{V}_2^*)$. Consequently the coefficients of these two quantities in the foregoing expression must vanish, and

$$\begin{aligned}\mathrm{Re}[Z_{21}] &= -\,\mathrm{Re}[Z_{12}] \\ \mathrm{Im}[Z_{21}] &= \mathrm{Im}[Z_{12}]\ .\end{aligned} \tag{27}$$

Conservative coupling thus implies that the "transfer impedances" satisfy

$$Z_{12} = -\,Z_{21}^*\ , \qquad |Z_{12}| = |Z_{21}|\ , \tag{28}$$

and in view of Eqs. (22) or (23), that the transfer admittances obey

$$|Y_{12}| = |Y_{21}| \quad . \tag{29}$$

Time-Average Modal Kinetic Energies

Since it is desired to develop relations which express the power flow between modes in terms of the time-average modal kinetic energies, it is of interest first to study these energies from a rather general viewpoint.

Clearly, the time-average value of the kinetic energy T_j of a mode j is given by

$$\langle T_j \rangle = \frac{m_j}{2} \langle V_j^2(t) \rangle \tag{30}$$

where m_j denotes the modal mass (or the mass of the system of which j is a mode) and V_j the modal velocity, and where the brackets $\langle \cdots \rangle$ indicate averaging with respect to time. The time-average value of the square of the modal velocity is also known as the mean-square modal velocity; it is independent of time, in the steady state. The problem of determining the average kinetic energies thus corresponds to that of finding the mean-square velocities, and these may be found from the velocities per se.

If the velocity of mass 1 of Fig. 1 due to a unit impulse force $\delta(t)$ acting on this mass is given by $y_{11}(t)$, then the velocity $V_1(t)$ due to a general force $F_1(t)$ acting on mass 1 may be expressed as a convolution (Refs. 1,6):

$$V_1(t) = \int_{-\infty}^{\infty} y_{11}(\alpha)\ F_1(t-\alpha)\ d\alpha \quad . \tag{31}$$

Here α is a dummy time variable, which disappears after the integration is performed and limits are substituted. Similarly, if the velocity of mass 1 due to a unit impulse force acting on mass 2 is given by $y_{12}(t)$, then the velocity of mass 1 due to a general force $F_2(t)$ acting on mass 2 may be written as

$$V_1(t) = \int_{-\infty}^{\infty} y_{12}(\beta)\ F_2(t-\beta)\ d\beta \quad , \tag{32}$$

where β is again a dummy variable.

According to the superposition principle, the response produced by a combination of excitations acting on a linear system is equal to the sum of the responses due to the individual excitations. Thus, the velocity $V_1(t)$ of mass 1 due to the combination of a force $F_1(t)$ acting on mass 1 and a force $F_2(t)$ acting on mass 2 may be expressed as

$$V_1(t) = \int_{-\infty}^{\infty} y_{11}(\alpha)\ F_1(t-\alpha)\ d\alpha + \int_{-\infty}^{\infty} y_{12}(\beta)\ F_2(t-\beta)\ d\beta \quad . \qquad (33)$$

The mean-square value of $V_1(t)$ is obtained by squaring the foregoing equation and averaging over all time.[3] If the two integrals of Eq. (33) are denoted by $J_{11}=J_{11}(t)$ and $J_{12}=J_{12}(t)$, respectively, one may write

$$\langle V_1^2 \rangle = \langle (J_{11}+J_{12})^2 \rangle = \langle J_{11}^2 \rangle + \langle J_{12}^2 \rangle + 2\langle J_{11}\ J_{12} \rangle \quad . \qquad (34)$$

The mixed product term may be expressed as

$$\langle J_{11}\ J_{12} \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_{11}(\alpha)\ y_{12}(\beta)\ \langle F_1(t-\alpha)\ F_2(t-\beta) \rangle\ d\alpha\ d\beta \ , \qquad (35)$$

where the averaging brackets have been placed only around those quantities which involve the variable t, with respect to which the averaging is to be performed. If F_1 and F_2 are uncorrelated, that is if

$$\int_{-\infty}^{\infty} F_1(t-\alpha)\ F_2(t-\beta)\ dt = 0 \qquad (36)$$

for all α, β, then $\langle F_1(t-\alpha)\ F_2(t-\beta) \rangle = 0$, and consequently $\langle J_{11}\ J_{12} \rangle = 0$.

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[3] In most cases one obtains an adequate approximation by averaging over many periods of the lowest frequency contained in $V_1(t)$.

The first term of Eq. (34) may be written analogously to Eq. (35) as

$$\langle J_{11}^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_{11}(\alpha)\, y_{11}(\beta)\, \langle F_1(t-\alpha)\, F_1(t-\beta) \rangle \, d\alpha \, d\beta \quad . \qquad (37)$$

Here the term within the averaging brackets may be recognized as the autocorrelation function R_1 of $F_1(t)$; this autocorrelation function obeys the relation (Ref. 6, pp. 20, 32-34)

$$R_1(\alpha-\beta) = \langle F_1(t-\alpha)\, F_1(t-\beta) \rangle = \langle F_1(t)\, F_1(t-\alpha+\beta) \rangle =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_1(\omega) e^{i\omega(\alpha-\beta)} \, d\omega \qquad (38)$$

where S_1 is defined as the mean square spectral density of the stationary random force $F_1(t)$ and is related to the autocorrelation function as

$$S_1(\omega) = \int_{-\infty}^{\infty} R_1(\tau) e^{-i\omega\tau} \, d\tau \quad . \qquad (39)$$

The functions $S_1(\omega)$ and $R_1(\tau)$ are a Fourier transform pair. So also are the impulse response and frequency response functions $y_{jk}(t)$ and $Y_{jk}(\omega)$, which for all j,k, obey[4]

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[4] It should be pointed out that several different valid Fourier transform definitions may be used. These differ only in the coefficients attached to the integrals, but all result in a factor of $1/2\pi$ for a complete transformation cycle. E.g., some authors use $1/\sqrt{2\pi}$ in each transformation. Here $1/2\pi$ has been used where the integrations are performed with respect to ω and unity where the integrations are performed with respect to τ.

It is also important to note that Eqs. (38)-(40) involve both positive and negative frequencies (and times), whereas only positive frequencies have physical meaning. An experimentally determined spectrum (involving only positive ω) may be converted to a spectrum corresponding to Eqs. (38) and (39) by halving the measured spectrum at each ω and plotting the result at both positive and negative numerical values of ω (Ref. 12, pp. 11,12).

$$y_{jk}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_{jk}(\omega)e^{i\omega t}\, d\omega \quad , \qquad Y_{jk}(\omega) = \int_{-\infty}^{\infty} y_{jk}(\tau)e^{-i\omega\tau}\, d\tau \tag{40}$$

$$Y^*_{jk}(\omega) = \int_{-\infty}^{\infty} y_{jk}(\tau)e^{i\omega\tau}\, d\tau.$$

Substitution of Eq. (38) into (37) gives

$$\begin{aligned}\langle J_1^2\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_{11}(\alpha)\, y_{11}(\beta)\, S_1(\omega)\, e^{i\omega(\alpha-\beta)}\, d\omega\, d\alpha\, d\beta \\ &= \int_{-\infty}^{\infty} S_1(\omega)\Big[\int_{-\infty}^{\infty} y_{11}(\alpha)\, e^{i\omega\alpha}\, d\alpha \int_{-\infty}^{\infty} y_{11}(\beta)\, e^{-i\omega\beta}\, d\beta\Big]\, d\omega \\ &= \int_{-\infty}^{\infty} S_1(\omega)\, Y^*_{11}(\omega)\, Y_{11}(\omega)\, d\omega = \int_{-\infty}^{\infty} S_1(\omega)|Y_{11}(\omega)|^2\, d\omega\ .\end{aligned} \tag{41}$$

Analogously one may write

$$\langle J_{12}^2\rangle = \int_{-\infty}^{\infty} S_2(\omega)\ |Y_{12}(\omega)|^2\, d\omega \tag{42}$$

where $S_2(\omega)$ denotes the mean-square spectral density of the force $F_2^2(t)$.

Substitution of the foregoing results into Eq. (34) permits one to write the mean-square velocity $\langle V_1^2\rangle$ of mass 1 as

$$\langle V_1^2\rangle = \int_{-\infty}^{\infty} S_1(\omega)|Y_{11}(\omega)|^2\, d\omega + \int_{-\infty}^{\infty} S_2(\omega)\ |Y_{12}|^2\, d\omega\ ; \tag{43}$$

and to set down by analogy the mean-square velocity of mass 2 as

$$\langle V_2^2 \rangle = \int_{-\infty}^{\infty} S_1(\omega) |Y_{21}|^2 \, d\omega + \int_{-\infty}^{\infty} S_2(\omega) |Y_{22}|^2 \, d\omega \quad . \qquad (44)$$

Calculation of Net Power Flow Between Modes by Superposition of Flows

In order to study the power flow (i.e., the rate of flow of energy) from one mode to another it is convenient to consider one force to act at a time, and then to add the power flows obtained in the two cases to obtain the net power flow. Such addition of power flows is permissible for statistically independent forces F_1 and F_2, as demonstrated below.

Since the system under consideration is linear, the force F_c exerted on mass 1 by the coupling element (or by the mass on the coupling element) obeys

$$F_c(t) = \int_{-\infty}^{\infty} h_{11}(\alpha_o) \, F_1(t-\alpha_o) \, d\alpha_o + \int_{-\infty}^{\infty} h_{12}(\beta_o) \, F_2(t-\beta_o) \, d\beta_o \qquad (45)$$

in analogy with Eq. (33). Here h_{11} and h_{12} represent impulse response functions; $h_{11}(t)$ denotes the coupling force $F_c(t)$ resulting from a unit impulse of force acting on mass 1, $h_{12}(t)$ similarly denotes the $F_c(t)$ resulting from such an impulse acting on mass 2, α_o and β_o are dummy time variables.

The time-average power P_{12} supplied to the coupling element may be obtained by multiplying the instantaneous force as given by Eq. (45) by the instantaneous velocity as given by Eq. (33), and averaging with respect to time. One obtains a result which one may write

$$P_{12} = \langle F_c(t) \, V_1(t) \rangle = P_{12a} - P_{21b} + \Delta P \qquad (46)$$

where

$$P_{12a} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{11}(\alpha_o)\, y_{11}(\alpha)\, \langle F_1(t-\alpha_o)\, F_1(t-\alpha)\rangle\, d\alpha_o\, d\alpha$$

$$P_{21b} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{12}(\beta_o)\, y_{12}(\beta)\, \langle F_2(t-\beta_o)\, F_2(t-\beta)\rangle\, d\beta_o\, d\beta \qquad (47)$$

$$\Delta P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{11}(\alpha_o)\, y_{12}(\beta)\, \langle F_1(t-\alpha_o)\, F_2(t-\beta)\rangle\, d\alpha_o\, d\beta$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{12}(\beta_o)\, y_{11}(\alpha)\, \langle F_2(t-\beta_o)\, F_1(t-\alpha)\rangle\, d\beta_o\, d\alpha \quad .$$

For conservative coupling, the time-average power flowing into the coupling element from mode 1 at steady state is exactly equal to the power flowing into mode 2, so that P_{12} of Eq. (46) represents also the power flow from mode 1 to mode 2. If $F_1(t)$ and $F_2(t)$ are uncorrelated, so as to satisfy Eq. (36), then the time averages appearing in the integrands of the foregoing expression for ΔP vanish and $\Delta P=0$. Since P_{12a} depends only on the excitation $F_1(t)$, it may be interpreted as the power flow from mode 1 to mode 2 due to the action of $F_1(t)$ by itself (i.e., with $F_2=0$). P_{21b} may be similarly interpreted, except that the minus sign included in its definition changes the direction considered that of positive power flow; P_{21b} thus represents the power flow from mode 2 to mode 1 due to the action of $F_2(t)$ by itself.

In view of the foregoing discussion one may thus write the net time-average (steady-state) power flow from mode 1 to mode 2 as

$$P_{12} = P_{12a} - P_{21b} \qquad (48)$$

where now P_{12a} denotes the flow from mode 1 to mode 2 if only mode 1 is excited, P_{21b} denotes the flow from mode 2 to mode 1 if only mode 2 is excited, and where the two excitations are taken to be uncorrelated.

One may calculate the power flow between modes directly by carrying out the integrations indicated in Eqs. (47), as Scharton has in essence done for a particular kind of coupling (Ref. 7). However, one may retain a greater degree of generality and present a discussion which is relatively easily understood by taking an approach based on Eq. (48) and on energy balance arguments, as in the following paragraphs.

General Expression for Power Flow

If only a force F_1 is present (that is, if $F_2=0$), then mass 2 is excited only through the coupling element, and in the steady state all of the power dissipated by the damping of system 2 must be supplied to this system from system 1 via the coupling element. An analogous statement applies if $F_1=0$ and $F_2\neq 0$. Let the time-average power flow from system j to system k be denoted by P_{jk}, and let the added subscript "a" designate quantities corresponding to the $F_2=0$ case, and subscript "b" to the $F_1=0$ case. In view of the previous energy balance arguments and of Eqs. (43) and (44) one may write[5]

$$\begin{aligned} P_{12a} &= c_2\langle V_{2a}^2\rangle = c_2 \int_{-\infty}^{\infty} S_1(\omega)\ |Y_{21}|^2\ d\omega \\ P_{21b} &= c_1\langle V_{1b}^2\rangle = c_1 \int_{-\infty}^{\infty} S_2(\omega)\ |Y_{12}|^2\ d\omega \end{aligned} \tag{49}$$

where S_1 and S_2 denote the mean square spectral densities of the forces F_1 and F_2. In view of Eqs. (48) and (49), the net power flow from system 1 to 2, when both external forces are acting, obeys the equation

$$\boxed{P_{12} = \int_{-\infty}^{\infty} [c_2\ S_1(\omega) - c_1\ S_2(\omega)]\ |Y_{12}|^2\ d\omega}\ . \tag{50}$$

[5]Note that $c_1 = \mathrm{Re}[Z_{11}]$ for the system of Fig. 1, if the coupling is conservative. In view of the definition of Z_{11}, as indicated following Eq. (22), Z_{11} is measured with $V_2=0$; hence no (dissipative) elements in system 2 can contribute terms to Z_{11}. Since the coupling is conservative it cannot contribute real terms to Z_{11}. Analogously, $c_2 = \mathrm{Re}[Z_{22}]$.

Relation Between Power Flow and Kinetic Energies

If F_1 and F_2 are "white noise" forces, that is if $S_1(\omega) = S_1$ and $S_2(\omega) = S_2$ are constant (independent of frequency), then one may introduce the notation

$$I_{jk} \equiv \int_{-\infty}^{\infty} |Y_{jk}|^2 \, d\omega \tag{51}$$

and rewrite Eqs. (33) and (34) as

$$\begin{aligned} \langle V_1^2 \rangle &= S_1 \, I_{11} + S_2 \, I_{12} \\ \langle V_2^2 \rangle &= S_1 \, I_{21} + S_2 \, I_{22} \end{aligned} \tag{52}$$

and Eq. (50) as

$$P_{12} = I_{12}(c_2 S_1 - c_1 S_2) \quad . \tag{53}$$

By solving Eqs. (52) for S_1 and S_2 in terms of $\langle V_1^2 \rangle$ and $\langle V_2^2 \rangle$ and introducing the result into Eq. (53), one obtains an expression for the power flow P_{12} in terms of the mean square velocities. By expressing the mean-square velocities $\langle V_j^2 \rangle$ in terms of the time-average kinetic energies $\langle T_j \rangle$ as indicated in Eq. (30), one finds that one may then write

$$P_{12} = \phi_{12} \langle T_1 \rangle - \phi_{21} \langle T_2 \rangle \tag{54}$$

where

$$\begin{aligned} \phi_{12} &\equiv \frac{2 \, I_{12} \left(\frac{c_2}{m_1} I_{22} + \frac{c_1}{m_1} I_{12} \right)}{I_{11} \, I_{22} - I_{12}^2} \\ \phi_{21} &\equiv \frac{2 \, I_{12} \left(\frac{c_1}{m_2} I_{11} + \frac{c_2}{m_2} I_{12} \right)}{I_{11} \, I_{22} - I_{12}^2} \end{aligned} \tag{55}$$

and where one has made use of the fact that $I_{12}=I_{21}$, as implied by Eq. (29).

Power Flow for Weak Coupling

The two systems of Fig. 1 may be said to be weakly coupled if the transfer admittance integrals $I_{12}=I_{21}$ have values which are much smaller than those of the "self-admittance" integrals I_{11} and I_{22}, and if the transfer impedances and admittances Z_{12}, Z_{21}, Y_{12}, Y_{21}, do not affect the values of I_{11} and I_{22} significantly.

Then, substitution of the first of Eqs. (23) into Eq. (51) yields

$$I_{11} = \int_{-\infty}^{\infty} |Y_{11}|^2 \, d\omega = \int_{-\infty}^{\infty} \left| \frac{Z_{22}}{Z_{11}Z_{22}-Z_{12}Z_{21}} \right|^2 d\omega \approx \int_{-\infty}^{\infty} |Z_{11}|^{-2} \, d\omega \tag{56}$$

$$\approx \int_{-\infty}^{\infty} \left| m_1(i\omega) + c_1 + k_1(i\omega)^{-1} \right|^{-2} d\omega = \frac{\pi}{c_1 m_1} \quad .$$

Note that the approximations indicated in Eq. (56) apply even if the transfer impedances and admittances are not strictly negligible (as compared to the self terms) over the entire frequency range, provided that the value of the integral is not significantly affected by contributions corresponding to transfer term values which are not negligible. For example, if Z_{11} and/or Z_{22} are small compared to Z_{12} and/or Z_{21} in certain frequency ranges (due to resonances of masses 1 and/or 2), then the approximations of Eq. (56) still hold, provided that these frequency ranges are narrow enough.

By use of the previously stated requirement $I_{12} \ll I_{11}$, I_{22} for loose coupling and of Eq. (56) one finds that one may restate Eqs. (54) and (55) as

$$\phi_{12} = \phi_{21} \approx \frac{2}{\pi} c_1 c_2 I_{12} \equiv \phi_c \quad , \tag{57}$$

$$\boxed{P_{12} = \phi_c(\langle T_1 \rangle - \langle T_2 \rangle)} \quad .$$

Power Flow for Stiffness and Gyroscopic Coupling

If the coupling element consists of an ideal stiffness (represented by a spring constant k_c) and an ideal gyroscopic element[6] (represented by a force/velocity coefficient c_g), one finds that

$$\begin{aligned} Z_{11} &= m_1(i\omega) + c_1 + (k_1+k_c)(i\omega)^{-1} \\ Z_{22} &= m_2(i\omega) + c_2 + (k_2+k_c)(i\omega)^{-1} \\ Z_{12} &= c_g - k_g(i\omega)^{-1} \\ Z_{21} &= - c_g - k_g(i\omega)^{-1} \quad . \end{aligned} \tag{58}$$

One may substitute these impedances into Eqs. (23) to find the various corresponding admittances, and one may then evaluate the integrals I_{11}, I_{22}, I_{12} exactly (by use of the theorem of residues, as summarized in Ref. 9, for example, or from the tabulated expression given on p. 72 of Ref. 6). After substitution of the results into Eqs. (55) and a considerable amount of algebraic manipulation one finds that the two coefficients ϕ_{12} and ϕ_{21} are here exactly equal, regardless of the magnitude of the coupling, and are given by

$$\phi_{12} = \phi_{21} = \phi_c = \frac{2}{m_1 m_2} \frac{k_c^2 \alpha_k + c_g^2 \alpha_c}{\alpha_k \alpha_c + (\omega_2^2 - \omega_1^2)^2} \quad , \tag{59}$$

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[6] A gyroscopic element is defined as one which produces a negative force on mass 2 due to a positive velocity of mass 1, if it results in a positive force on mass 1 due to a positive velocity of mass 2. (See Ref. 8.) Note that a viscous damping element results in forces of the same sign on both masses due to positive velocities, and that gyroscopic coupling is conservative since the Z_{12} and Z_{21} of Eqs. (58) satisfy the requirement of Eq. (28).[12]

where

$$\alpha_k \equiv \frac{c_1}{m_1} + \frac{c_2}{m_2} = 2(\xi_1\omega_1 + \xi_2\omega_2)$$

$$\alpha_c \equiv \frac{c_1k_2 + c_2k_1}{m_1m_2} = 2\,\omega_1\omega_2(\xi_1\omega_2 + \xi_2\omega_1) \qquad (60)$$

$$\omega_1^2 \equiv (k_1+k_c)/m_1 \quad , \quad \omega_2^2 \equiv (k_2+k_c)/m_2$$

$$\xi_1 \equiv 2\,m_1\omega_1 \quad , \quad \xi_2 \equiv 2\,m_2\omega_2 \quad .$$

The definitions of Eqs. (60) were chosen so that ω_j represents the natural frequency of the j^{th} mass (or mode) if the other mass is kept from moving, and so that ξ_j denotes the ratio of the damping coefficient c_j to the critical damping coefficient $2\,m_j\omega_j$ of the j^{th} mass.

The foregoing result is found to agree with that which Scharton (Ref. 7) derived for pure spring coupling by calculating explicitly the power flow through the coupling element. For small coupling the foregoing result also reduces to that which was obtained by Lyon and Maidanik (Ref. 10) by means of an explicit power flow calculation based on small coupling approximations.

Finite Frequency Band Excitation

The previous discussion was based on force spectra S_1 and S_2 that are assumed constant for all frequencies. The results obtained there similarly apply to power flow and system energies involving the entire infinite frequency band. However, a little reflection permits one to interpret these results also for cases where only a finite frequency band is of interest.

One may obtain expressions pertaining to only a finite frequency band by replacing the infinite limits appearing in the integrals of Eqs. (49)-(51) by the frequencies limiting the band. Equations (52)-(55) then still apply, although the various I_{jk} integrals that enter them now are different.

The magnitude of the admittance $Y(\omega)$ of a not too highly damped single-degree-of-freedom system is much greater near the system natural frequency (resonance) than at other frequencies, so that the greatest contribution to the value of the integral $I = \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$ is due to the frequency region near the system resonance. Consequently, the value of the integral will be changed only slightly if the infinite limits are replaced by finite ones, provided that the interval of integration (i.e., the frequency band under consideration) still encompasses the system resonance. (See, for example, Ref. 11, or pp. 21ff, 29ff of Ref. 5.)

Similar reasoning applied to a two-degree-of-freedom system (made up of two coupled single-degree-of-freedom systems), which has two resonances, leads one to conclude that the results of Eqs. (56), (57), (59) and (60) also are good approximations if the excitation frequency bands are finite, but encompass both system resonances.

In view of the previously discussed predominance of the responses near the resonances, the aforementioned results may also be taken to be good approximations for cases where the excitation spectra are not flat. All that is required for these approximations to hold is that the excitation spectra exhibit no violent peaks; i.e., no peaks that are of the same order as the peaks in the $|Y|^2$ curves.

Summary and Conclusions

It has been shown in this section that the steady-state time-average power flow from one mode to another is proportional to the difference between the time-average kinetic energies of the two modes, provided that both

1) the coupling between the two modes is
   - linear (giving rise to a linear differential equation),
   - conservative (neither supplying nor extracting mechanical energy), and
   - light and/or purely spring-like and/or gyroscopic;

and

2) the forces acting on the two modes are uncorrelated and have spectra that are relatively flat (as compared to the system admittance spectra) within the frequency band encompassed by the resonances of the coupled system.

Expressions which give the constant of proportionality ϕ_c between power flow and kinetic energy difference in terms of the characteristics of the modes and of the coupling elements have been derived. These expressions are given in Eqs. (57) and (59).

Since the square of the difference between the squares of the modal resonance frequencies appears in the denominator of Eq. (59), one may note that the proportionality constant ϕ_c decreases rapidly as the frequency difference increases. Thus, the power flow between two modes whose resonance frequencies are nearly the same will be much greater than the flow between similar modes whose resonances are widely separated.

The relation between power flow and kinetic energies, which has been established in this section, permits one to analyze the average dynamic behavior of coupled modes rather simply on the basis of energy balance considerations. For such energy analyses it is convenient to visualize coupled modes as shown in Fig. 2. There the modes are represented as storage elements (containing kinetic energies $\langle T_1 \rangle$ and $\langle T_2 \rangle$), connected by power flow P_{12}, supplied with power inputs A_1 and A_2 from sources external to the system, and experiencing power losses (dissipations) D_1 and D_2, respectively.

POWER FLOW BETWEEN TWO SETS OF MODES

As has been discussed earlier, one may analyze the vibrations of an elastic system in terms of the vibrations of the set of the modes of the system. The interaction between two elastic systems may correspondingly be studied by investigating the interaction between two sets of modes which represent the two systems.

The modes of an ideal system vibrate independently of each other, as has been pointed out previously, and the modes of lightly damped realistic systems tend to behave similarly. Hence, the present section deals with the power flow between two sets of modes, where the members of each set are not coupled to each other, but are coupled to all members of the other set.

The problem discussed in the present section may be visualized with the aid of Fig. 3, which schematically shows two sets of modes, one designated by "α", the other by "β". It is assumed that no power flow occurs between modes which are members of the same set[7], but that each mode of the α set is coupled to each mode of the β set. It is desired to determine the flow of energy from the α set to the β set in terms of the previously studied mode-to-mode flows, under conditions where each of the modes of each set may dissipate energy and may be supplied with energy from outside the system (as indicated in Fig. 3).

In particular, it is desired to determine under what conditions one may obtain relations for the <u>total net</u> power flow between two <u>sets</u> of modes in the form

$$\boxed{P_{\alpha\beta} = \phi_{\alpha\beta}\, T_\alpha - \phi_{\beta\alpha}\, T_\beta} \quad , \tag{61}$$

in analogy to Eq. (54), which pertains to power flow between two <u>single</u> modes. Here T_α and T_β denote the <u>set-averages</u> of the time-average modal kinetic energies, which are discussed in more detail in the subsequent paragraphs. Moreover, one desires to establish for coupled sets a power flow diagram like Fig. 4, in analogy to the power flow diagram of Fig. 2, which pertains to two coupled modes.

Throughout the following discussion it is assumed that the power flow between each pair of coupled modes is proportional to the difference between the time-average modal kinetic energies, as indicated by the second of Eqs. (57). It is convenient for the present purposes to rewirte the aforementioned expression as

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[7]No power flow between these modes occurs either if these modes are not coupled to each other, or if all of these modes contain equal kinetic energies. See Eq. (57).

$$P_{\alpha j,\beta k} = \phi_{\alpha j,\beta k}(T_{\alpha j} - T_{\beta k}) \quad . \tag{62}$$

Here $P_{\alpha j,\beta k}$ denotes the power flow from the j^{th} mode of the α set to the k^{th} mode of the β set, and $T_{\alpha j}$ and $T_{\beta k}$ denote the time-average kinetic energies of these two modes (due to effectively white-noise excitation), in agreement with Eq. (57). In Eq. (62) and subsequently the brackets previously used to designate time-averages are omitted for the sake of economy of notation; however, all kinetic energy and power flow quantities appearing here and henceforth are intended to represent time-average values.

Power Flow Between Mode Sets in Terms of Set-Average Energies

The total power flow $P_{\alpha\beta}$ from mode set α to mode set β is the sum of all the individual mode-to-mode power flows. If set α contains N_α modes and set β contains N_β modes, then one may express the set-to-set power flow as

$$P_{\alpha\beta} = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} P_{\alpha j,\beta k} = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k}(T_{\alpha j} - T_{\beta k}) \tag{63}$$

in view of Eq. (62).

The set-average modal kinetic energies T_α and T_β of the two sets are defined as

$$T_\alpha \equiv \frac{1}{N_\alpha} \sum_{j=1}^{N_\alpha} T_{\alpha j} \quad , \qquad T_\beta \equiv \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} T_{\beta k} \quad . \tag{64}$$

If Eqs. (61) and (63) are to agree, then the relation

$$\sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k}(T_{\alpha j} - T_{\beta k}) = \frac{\phi_{\alpha\beta}}{N_\alpha} \sum_{j=1}^{N_\alpha} T_{\alpha j} - \frac{\phi_{\beta\alpha}}{N_\beta} \sum_{k=1}^{N_\beta} T_{\beta k} \tag{65}$$

must hold for all values of $T_{\alpha j}$ and $T_{\beta k}$. Consequently the set-to-set coupling factors must obey

$$\phi_{\alpha\beta} = \left[N_\alpha \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k}\, T_{\alpha j} \right] \div \left[\sum_{j=1}^{N_\alpha} T_{\alpha j} \right]$$

$$\phi_{\beta\alpha} = \left[N_\beta \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k}\, T_{\beta k} \right] \div \left[\sum_{k=1}^{N_\beta} T_{\beta k} \right] \qquad (66)$$

and the equivalent relations

$$\phi_{\alpha\beta} = N_\alpha N_\beta \left\{ \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} \left[\left(\sum_{j=1}^{N_\alpha} \phi_{\alpha j,\beta k}\, T_{\alpha j} \right) \div \left(\sum_{j=1}^{N_\alpha} T_{\alpha j} \right) \right] \right\}$$

$$\phi_{\beta\alpha} = N_\alpha N_\beta \left\{ \frac{1}{N_\alpha} \sum_{j=1}^{N_\alpha} \left[\left(\sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k}\, T_{\beta k} \right) \div \left(\sum_{k=1}^{N_\beta} T_{\beta k} \right) \right] \right\}. \qquad (67)$$

The latter expressions have an interesting interpretation. The term within the large square brackets of the first of the foregoing equations is the weighted average of the $\phi_{\alpha j,\beta k}$ coefficients (for a given value of k), with the modal energies $T_{\alpha j}$ acting as weighting factors. The term within the curved brackets may be seen to be the average (over all values of k) of the aforementioned weighted averages. Thus, $\phi_{\alpha\beta}$ is $N_\alpha N_\beta$ times the average value of $\phi_{\alpha j,\beta k}$ (in the previously indicated sense); this implies that the power flow between mode sets is equal to the average mode-to-mode flow times the number of mode pairs between which power flow occurs.

Modal Energy-Independent Coupling Factors

If the set-to-set coupling factors $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$ depend on the modal energies $T_{\alpha j}$ and $T_{\beta k}$ as in Eqs. (66) and (67), then use of these factors in Eq. (61) does not simplify the calculation of power flow between mode sets. However, in some cases the coupling factors do turn out to be independent of the energies $T_{\alpha j}$ and $T_{\beta k}$'s, and then Eq. (61) does present a convenient means for calculating the set-to-set power flows. Conditions under which the coupling factors tend to be independent of the energies are explored in the present section.

Uniform Coupling

For the case where all of the mode-to-mode coupling factors are equal, that is, where $\phi_{\alpha j,\beta k} = \phi_o$ for all j and k, one finds that the set-to-set coupling coefficients of Eqs. (66) or (67) are independent of the modal kinetic energies and are given by

$$\boxed{\phi_{\alpha\beta} = \phi_{\beta\alpha} = N_\alpha N_\beta \, \phi_o} \quad . \tag{68}$$

One may observe, as also previously mentioned, that here the set-to-set coupling coefficient is $N_\alpha N_\beta$ times the mode-to-mode coefficient, and that $N_\alpha N_\beta$ is the number of mode-to-mode paths along which power can flow from members of one set to members of the other.

Equation (57) indicates that the assumption of uniform coupling may be useful under some often-encountered practical circumstances. If all modes of set "α" are similar to each other (i.e., have nearly the same resonance frequency and the same amount of damping), if all modes of set "β" are similar to each other, and if all α modes are similarly coupled to all β modes (i.e., if all mode-to-mode coupling is characterized by the same kinds and magnitudes of coupling elements), then all the mode-to-mode coupling factors will be the same. Thus, Eq. (68) applies to mode sets that satisfy the foregoing similarity criteria. Indeed, an essential step in the practical application of the statistical energy approach consists of organizing the set of all modes of a system into subsets within which all modes have similar characteristics, so that relations like Eq. (68) can be applied to these subsets.

Uniform Modal Energies

If all of the modal kinetic energies in set "α" are equal, and if the same is true of set "β", so that one may write $T_{\alpha j}=T_{\alpha o}$ for all j and $T_{\beta k}=T_{\beta o}$ for all k, then Eqs. (66) or (67) are found to reduce to

$$\phi_{\alpha\beta} = \phi_{\beta\alpha} = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \phi_{\alpha j,\beta k} \quad , \tag{69}$$

This expression indicates that here the set-to-set coupling factor is equal to the sum of the mode-to-mode factors for all coupled mode pairs. (This sum clearly is equal to $N_\alpha N_\beta$ times the average value of the mode-to-mode coupling factor between all "α" and "β" modes.)

In order to investigate the circumstances under which the assumption of uniform modal energies is a realistic one, one needs to study the general modal kinetic energy $T_{\beta k}$. An expression for this energy may be obtained from a steady-state time-average energy balance relation for the $k^{\underline{th}}$ mode of set "β", which relation may be written as

$$A_{\beta k} + \sum_{j=1}^{N_\alpha} P_{\alpha j,\beta k} = D_{\beta k} \quad , \tag{70}$$

or

$$A_{\beta k} + \sum_{j=1}^{N_\alpha} \phi_{\alpha j,\beta k}(T_{\alpha j}-T_{\beta k}) = c_{\beta k} \frac{2}{m_{\beta k}} T_{\beta k} \quad . \tag{71}$$

As indicated in Fig. 3, $A_{\beta k}$ represents the power supplied to the mode directly from external sources, $D_{\beta k}$ denotes the power dissipated by the mode (which depends on the modal viscous damping coefficient $c_{\beta k}$ and on the mean square velocity, and thus on the modal kinetic energy $T_{\beta k}$), and $P_{\alpha j,\beta k}$ denotes the power supplied from the $j^{\underline{th}}$ mode of the "α" set to the mode under consideration.

Solution of Eq. (71) for $T_{\beta k}$ yields

$$T_{\beta k} = \left(A_{\beta k} + \sum_{j=1}^{N_\alpha} \phi_{\alpha j, \beta k} \, T_{\alpha j} \right) \div \left(\frac{2c_{\beta k}}{m_{\beta k}} + \sum_{j=1}^{N_\alpha} \phi_{\alpha j, \beta k} \right) , \quad (72)$$

which expresses the kinetic energy $T_{\beta k}$ of the single mode βk in terms of the modal and coupling parameters and of the energies $T_{\alpha j}$ of all of the modes of the "α" set. One may use Eq. (72) to investigate under what conditions all of the "β" modes possess approximately equal time-average kinetic energies.

If those terms of Eq. (72) which contain the coupling factors are negligible compared to the others, then this equation reduces to

$$T_{\beta k} \approx A_{\beta k} \, m_{\beta k}/2 \, c_{\beta k} \quad . \quad (73)$$

For negligible coupling the power input may be shown to be given by[8]

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[8]If coupling effects are negligible, then the power input is equal to the power dissipation, or (omitting all subscripts since only one system is being considered here)

$$A = D = c\langle V^2 \rangle = cSI = \pi S/m \quad ,$$

in view of Eqs. (52) and (56).

One may verify that one may obtain the same result by direct calculation of the power input:

$$A = \langle F(t) \, V(t) \rangle = \int_{-\infty}^{\infty} y(\alpha) \, \langle F(t-\alpha) \, F(t) \rangle \, d\alpha \quad .$$

Since the force autocorrelation function $R_f(\alpha)$ is related to the spectrum S of a white noise force (Ref. 12) as

$$\langle F(t-\alpha) \, F(t) \rangle = R_f(\alpha) = 2\pi \, S_a \, \delta(\alpha) \quad ,$$

where $\delta(\alpha)$ is the delta or unit impulse function, the above reduces to

$$A = 2\pi S \int_{-\infty}^{\infty} y(\alpha) \, \delta(\alpha) \, d\alpha = \pi S [y(0+) + y(0-)] = \pi S/m \quad ,$$

since $y(0+) = 1/m, \quad y(0-) = 0.$

$$A_{\beta k} = \pi \, S_{\beta k}/m_{\beta k} \quad , \tag{74}$$

where $S_{\beta k}$ denotes the spectral density of the force which acts directly on the βk mode. One may then rewrite Eq. (74) as

$$T_{\beta k} \approx (\pi/2)(S_{\beta k}/c_{\beta k}) \quad . \tag{75}$$

This result indicates that in the presence of small coupling the kinetic energies of all "β" modes will be equal if the excitations are adjusted so that the ratio of modal force spectral density to modal damping is the same for all modes. As a special case, if all modes of the β set are equally excited and equally damped, then they will have equal kinetic energies.

Some other observations relevant to uniform distribution of energies may be made from an analysis of indirectly excited modes. If the modes of the "β" set are lightly damped and not directly excited ($A_{\beta k}=0$) then the net power flow to the $j^{\underline{th}}$ mode from all the modes of the "α" set must be very small, and one may determine from Eq. (71) or Eq. (72) that

$$T_{\beta k} \approx \left(\sum_{j=1}^{N_\alpha} \phi_{\alpha j, \beta k} \, T_{\alpha j} \right) \div \left(\sum_{j=1}^{N_\alpha} \phi_{\alpha j, \beta k} \right) \tag{76}$$

Equation (76) shows $T_{\beta k}$ here to be a weighted average of all of the $T_{\alpha j}$'s (with the coupling coefficients $\phi_{\alpha j, \beta k}$ acting as weighting factors). One notes that the kinetic energies $T_{\beta k}$ of all of the "β" modes will be equal if the sums appearing in Eq. (76) are independent of the index k, or if all of the "α" modes have the same kinetic energy $T_{\alpha j}$. Thus, if the total coupling to all "β" modes is uniform, and/or if the energy distribution in the "α" set is uniform, then the energy distribution in the indirectly excited "β" set will also be uniform.

Equation (76) also implies that the distribution of energies $T_{\beta k}$ in the indirectly excited "β" set will generally be smoother than that of the energies $T_{\alpha j}$ in the "α" set, if the summations do not depend very strongly on the index k. In other words, the $T_{\beta k}$'s will approach equality to each other more closely than do the $T_{\alpha j}$'s, provided that the coupling of the "β" modes to all of the "α" modes is about the same for all of the "β" modes.

One may thus visualize the behavior of a cascade of lightly damped mode sets, where only the first set is directly excited, where each set is relatively loosely coupled to its neighbors, and where the mode-to-mode coupling for each pair of adjacent sets is more or less uniform. (This does not imply that the mode-to-mode coupling need have the same value for all set pairs.) If the first set of such a cascade is excited so that an arbitrary distribution of kinetic energies is produced in it, then the second set will attain an energy distribution which is more uniform than that in the first, in accordance with the previous paragraph. Similarly, the energy distribution in the third set will then be more uniform than that in the second, and so on.

Power Flow in the Ensemble Average

In practice one rarely deals with structures whose every detail is precisely known. For example, manufacturing tolerances produce geometric differences, and variations in material properties result in deviations from ideal homogeneity and isotropy. Hence, one may expect to encounter some differences in the properties and distributions of modes in nominally identical structures, and thus in the responses of these structures to the same excitation. In order to study the behavior of "average" structures, one is led to consider the power flow that may be expected to occur "on the average" between two elastic systems, where each of the two is a member of an ensemble, and where the members of each ensemble can differ from each other to some degree.

Instead of dealing with only a single "α" set and a single "β" set of modes (corresponding to a single α elastic system and a single β elastic system), it is necessary here to deal with collections or "ensembles" of such sets (corresponding to ensembles of elastic systems). The problem to be considered here consists of determining the ensemble average of the time-average power flow between α and β mode sets, in terms of the ensemble average of the time-average kinetic energies.

Consider an "experiment", where one selects at random from the ensembles one set of the type α and one of the type β, subjects each of the selected pair of sets to a specified excitation, and measures the time-average power flow and kinetic energies. It is desired to express the average power flow measured in a number of such experiments in a form analogous to Eq. (61), as

$$\overline{P}_{\alpha\beta} = \tilde{\phi}_{\alpha\beta}\,\overline{T}_{\alpha} - \tilde{\phi}_{\beta\alpha}\,\overline{T}_{\beta} \quad , \tag{77}$$

where the bar denotes the average of the values obtained in these experiments (i.e., the ensemble average) and the tilde (~) is intended to differentiate the present coupling coefficients from those of Eq. (61), which pertain to two fully defined mode sets; i.e. to a single "experiment".

One may similarly modify Eqs. (62)-(66) to apply to ensemble averages and to the coupling coefficients as defined by Eq. (77) and arrive at expressions analogous to Eqs. (67). Noting that one may interchange the order of summation and averaging (which essentially is another summation), one obtains

$$\tilde{\phi}_{\alpha\beta} = N_{\alpha}N_{\beta}\left\{\frac{1}{N_{\beta}}\sum_{k=1}^{N_{\beta}}\left[\left(\sum_{j=1}^{N_{\alpha}}\overline{\phi_{\alpha j,\beta k}\,T_{\alpha j}}\right) \div \left(\sum_{j=1}^{N_{\alpha}}\overline{T_{\alpha j}}\right)\right]\right\} \tag{78}$$

$$\tilde{\phi}_{\beta\alpha} = N_{\alpha}N_{\beta}\left\{\frac{1}{N_{\alpha}}\sum_{j=1}^{N_{\alpha}}\left[\left(\sum_{k=1}^{N_{\beta}}\overline{\phi_{\alpha j,\beta k}\,T_{\beta k}}\right) \div \left(\sum_{k=1}^{N_{\alpha}}\overline{T_{\beta k}}\right)\right]\right\} .$$

The average of the product of two quantities is not in general equal to the product of the averages; one cannot in general replace $\overline{\phi_{\alpha j,\beta k}\,T_{\alpha j}}$ by $(\overline{\phi_{\alpha j,\beta k}})(\overline{T_{\alpha j}})$, as one would like to do in order to simplify Eqs.(78). However, the aforementioned interchange of the averaging and multiplication processes is valid for random variables that are statistically independent. This means that $\overline{\phi_{\alpha j,\beta k}\,T_{\alpha j}} = (\overline{\phi_{\alpha j,\beta k}})(\overline{T_{\alpha j}})$, provided that the probability of obtaining any value of $T_{\alpha j}$ in a given "experiment" is unaffected by the value of $\phi_{\alpha j,\beta k}$ in that experiment, and vice versa.

The various modal kinetic energies which are produced by a given excitation distribution do depend on the various mode-to-mode coupling factors (as one can readily determine from any energy balance analysis), and thus cannot be entirely statistically independent of the coupling. However, it is likely that a given $T_{\alpha j}$ will depend only very weakly on a single $\phi_{\alpha j,\beta k}$, particularly if many modes are present in the two sets; then these two quantities may be very nearly independent. If the excitation distribution may also vary from experiment to experiment (e.g., in the relatively realistic case where the modal excitations depend on the sample structure selected for an "experiment") then statistical independence is even more likely.

For statistically independent mode-to-mode coupling coefficients and modal kinetic energies one may replace $\phi_{\alpha j,\beta k}\, T_{\alpha j}$ by $(\overline{\phi_{\alpha j,\beta k}}) \cdot \overline{T_{\alpha j}}$ in the first of Eqs. (78) and perform a similar replacement in the second of these equations. If $\overline{T_{\alpha j}}$ is independent of the mode number j (that is, if all modes of set α have the same kinetic energy in the ensemble average) and if a similar statement holds about the β modes, then one obtains from Eqs. (78) the simple result

$$\boxed{\tilde{\phi}_{\alpha\beta} = \tilde{\phi}_{\beta\alpha} = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{N_\beta} \overline{\phi_{\alpha j,\beta k}}} \quad . \tag{79}$$

Equation (79), which is analogous to Eq. (69), indicates that the coupling coefficients to be used in Eq. (77) are equal to the sum of the ensemble average (or the ensemble average of the sum) of the mode-to-mode coupling factors. Of course, Eq. (79) applies only if the conditions used in its derivation are met, that is, if the modal kinetic energies and mode-to-mode coupling factors are statistically independent, and if the ensemble average modal kinetic energies are independent of mode number (constant within a type of set).

Summary and Conclusions

In this section it has been shown that the steady-state time-average power flow from one set of modes to another is proportional (or approximately proportional) to the difference between the set-average modal kinetic energies of the two sets, provided that both

1) the mode-to-mode coupling satisfies the conditions discussed under the heading of "Power Flow Between Two Modes", so that the power flows between modes of the two sets may be taken as proportional to the modal kinetic energy differences;

and

2) either a) the mode-to-mode coupling is (at least approximately) the same for all mode pairs;

   or b) all modes in a set have (at least approximately) equal time-average kinetic energies.

Then the factor of proportionality; i.e., the coupling factor, applicable to set-to-set power flow is equal to the sum of all the mode-to-mode coupling factors (or to the product of the number of mode pairs and of the average value of all of the individual mode-to-mode coupling factors).

It has also been shown that the ensemble average of the steady-state time average power flows from one set of an ensemble to a set of another ensemble is proportional to the difference between the ensemble average of the set average modal kinetic energies of the two sets, - provided (1) that the first of the previously listed conditions is satisfied, (2) that the mode-to-mode coupling coefficients and modal kinetic energies are statistically independent, and (3) that the modal kinetic energies in the ensemble average are uniformly distributed within a set. The applicable coupling factor then is the ensemble average of the sum (or the sum of the ensemble average) of all the individual mode-to-mode coupling factors.

The mode-to-mode coupling is essentially the same for all mode pairs if all modes of the α set are similar to each other, if all modes of the β set are similar to each other, and if the same (or similar) physical mechanisms (e.g., the same type of sound-to-structure coupling) couples all modes of the α set to all modes of the β set.

All modes in a set have equal kinetic energies (i.e., "equipartition of energy" occurs among the modes of a set), if

either a) the modes of the set are uniformly excited and either weakly or uniformly coupled to modes of other sets;

or b) the modes are lightly damped, not excited directly, and uniformly coupled to a set of modes within which all modes have nearly the same energy.

In the special cases discussed in this section, and only in these cases, the average response and dynamic interaction of mode sets can be reduced to simple terms, represented schematically in Fig. 4. The most important expressions pertaining to these special cases are also summarized in this figure. In analyzing the total interaction of two systems by means of the statistical energy approach, one must generally classify the modes of the two systems into sets, such that the modes within each set are sufficiently similar so as to satisfy the conditions which permit this simple treatment of set interaction to be applied.

LOSS FACTORS, RESISTANCES, AND MODAL DENSITIES

The foregoing development of the concepts of the statistical energy method was presented in terms of power flow coupling coefficients and in terms of the number of modes in a set. These parameters are relatively unfamiliar, and generally more difficult to estimate, calculate, or measure than such more commonly used quantities as loss factors, coupling resistances, and modal densities. It is the purpose of the present section to present the relations between these more common parameters and those used in the previous sections, so that one may restate the previously derived results in terms of the more familiar quantities.

The relations between these two sets of parameters are simple, so that restatement of the previous results in terms of the more common quantities involves only trivial algebraic manipulations. Hence, no restated results are given here. Some applications of the relations given in the present section are illustrated in the subsequent one.

Loss Factors

The loss factor is classically defined on the basis of a single-degree-of-freedom system oscillating sinusoidally at resonance in the steady state. The definition of the loss factor η may be stated as

$$\eta \equiv E_d/2\pi E \quad , \tag{80}$$

where E_d denotes the energy dissipated per cycle by the system and E denotes the "energy of vibration", which is equal essentially to the total (kinetic plus potential) energy at any time (and is nearly constant), to the time-wise maximum kinetic energy, and also to the time-wise maximum potential energy of the system (Refs. 13, 14).

One may express the loss factor in terms of the time-average power D dissipated by the system, since $D = E_d \omega_o/2\pi$, where ω_o denotes the radian natural frequency of the system so that $\omega_o/2\pi$ represents the natural frequency in cycles/time. One obtains

$$\eta = \frac{D}{\omega_o E} = \frac{D}{2\omega_o \bar{T}} \tag{81}$$

where T denotes the time-average kinetic energy and T=E/2 holds for the postulated oscillations. Since for a simple mass-spring-dashpot system $D = c\langle v^2 \rangle$ and $T = \langle v^2 \rangle m/2$, one may readily verify that for such a system

$$\eta = \frac{c}{m\omega_o} = \frac{c}{\sqrt{km}} = 2\,\frac{c}{c_c} \quad , \tag{82}$$

where c_c denotes the critical viscous damping coefficient of the system. Equation (82) involves the viscous damping coefficient c, and hence applies only for damping of the viscous type; Eq. (82) thus is less general than Eqs. (80) and (81), which are not restricted to any particular type of energy dissipation mechanism. Since the loss factor applies for all damping mechanisms, it is generally preferable to more restricted measures of damping.

A broadened interpretation of Eq. (81) is often used to define (and measure) loss factors. In this interpretation ω_o is taken as the center frequency of a band encompassing the system natural frequency, E is taken as the system energy in that band, and D is taken as the power dissipated by or (transferred away from) the system in the same band. This interpretation is generally not needed for dealing with a single-degree-of-freedom system, but is useful when one is concerned with a collection of such systems, - i.e., with a set of modes. For a set of modes, D of Eq. (81) is taken as the total power dissipated by all modes and T as total kinetic energy of all modes. For a set α containing N_α individual modes, the total power dissipation $D_{T\alpha}$ and total time-average kinetic energy $T_{T\alpha}$ are given by

$$D_{T\alpha} = \sum_{j=1}^{N_\alpha} D_{\alpha j} = 2\omega_o \sum_{j=1}^{N_\alpha} \eta_{\alpha j}\, T_{\alpha j} \quad ,$$

$$T_{T\alpha} = \sum_{j=1}^{N_\alpha} T_{\alpha j} = N_\alpha\, T_\alpha \tag{83}$$

where $\eta_{\alpha j}$ denotes the loss factor of the αj mode, as defined by Eq. (81), and T_α denotes the set-average kinetic energy, as in Eq. (64). The loss factor η_α of set α then obeys

$$\eta_\alpha \equiv \frac{D_{T\alpha}}{2\omega_o T_{T\alpha}} = \frac{D_{T\alpha}}{2\omega_o N_\alpha T_\alpha} \tag{84}$$

$$= \left(\sum_{j=1}^{N_\alpha} \eta_{\alpha j} T_{\alpha j} \right) \div \left(\sum_{j=1}^{N_\alpha} T_{\alpha j} \right) ,$$

where now ω_o is defined as the center frequency of a band which encompasses all modal resonances. The last form of the above equation indicates that the set loss factor η_α is a weighted average of the individual modal loss factor $\eta_{\alpha j}$, with the modal kinetic energies $T_{\alpha j}$ acting as weighting factors.

Coupling Loss Factors

Mechanical energy may be lost from a given system not only by being dissipated (i.e., being changed to thermal energy), but also by being transferred to another system. Energy conducted from a system is lost to it, just like energy dissipated by it. Hence, it is useful to define a loss factor for energy conduction analogous to that for energy dissipation.

It is convenient to define a coupling loss factor $\eta_{\alpha j,\beta k}$ of mode αj to describe the time-average power flow $P_{\alpha j,\beta k}$ from mode αj to mode βk in analogy to Eq. (81) by

$$\eta_{\alpha j,\beta k} \equiv \left. \frac{P_{\alpha j,\beta k}}{2\ \omega_o\ T_{\alpha j}} \right|_{T_{\beta k} = 0} . \tag{85}$$

The condition $T_{\beta k}=0$ is imposed in order to make this loss factor independent of the modal kinetic energies. In view of Eq. (62), one finds that the coupling loss factor is related to the mode-to-mode coupling factor $\phi_{\alpha j,\beta k}$ as

$$\eta_{\alpha j,\beta k} = \phi_{\alpha j,\beta k} / 2\ \omega_o . \tag{86}$$

Eq. (86) indicates that the same coupling loss factor applies for power flow in either direction (that is, $\eta_{\alpha j,\beta k} = \eta_{\beta k,\alpha j}$) for those various previously discussed cases in which the same coupling factor applies for power flow in either direction ($\phi_{\alpha j,\beta k} = \phi_{\beta k,\alpha j}$).

One may also define an analogous loss factor to account for power flow from one set of modes (α) to another (β). The coupling loss factor $\eta_{\alpha\beta}$ of set α is defined as

$$\eta_{\alpha\beta} \equiv \left.\frac{P_{\alpha\beta}}{2\omega_o \, T_{T\alpha}}\right|_{T_\beta=0} , \tag{87}$$

where ω_o denotes the center frequency of a band which encompasses all modal resonances of both sets. In view of Eqs. (61) and (83), one may rewrite the foregoing as

$$\eta_{\alpha\beta} = \frac{\phi_{\alpha\beta} \, T_\alpha}{2\omega_o \, N_\alpha \, T_\alpha} = \frac{\phi_{\alpha\beta}}{2 \, \omega_o \, N_\alpha} . \tag{88}$$

Here it is important to note that because of the appearance of N_α in Eq. (88) the coupling loss factor $\eta_{\beta\alpha}$ of set β (for power flow from β to α) is generally not the same as the loss factor $\eta_{\alpha\beta}$ of set α (for power flow from α to β), even if the coupling coefficients $\phi_{\alpha\beta} = \phi_{\beta\alpha}$ are the same for both flow directions (as in the previously analyzed cases). Instead, in view of Eq. (88) and an expression analogous to it,

$$\eta_{\alpha\beta} \, N_\alpha = \eta_{\beta\alpha} \, N_\beta = \phi_{\alpha\beta}/2 \, \omega_o . \tag{89}$$

Loss and Coupling Resistances

In analogy to electrical circuits, in which the time-average power dissipation is given by $\langle I^2 \rangle R_e$, where I denotes current and R_e electrical resistance, one may define a mechanical or acoustic resistance R of an elastic system so that the power loss D from the system is given by

$$D = R \langle V^2 \rangle . \tag{90}$$

For a viscously damped single-degree-of-freedom system the resistance is equal to the viscous damping coefficient c, of course. In view of Eq. (82) one may relate this resistance to the loss factor η and obtain

$$R = c = \eta \, m \, \omega_o \tag{91}$$

for a system of mass m and natural frequency ω_o.

One may define a coupling resistance $R_{\alpha j, \beta k}$ which describes the power loss of a mode α_j to a mode βk analogously to Eq. (90) by

$$R_{\alpha j, \beta k} \langle V_{\alpha j}^2 \rangle = P_{\alpha j, \beta k} \Big|_{T_{\beta k} = \langle V_{\beta k}^2 \rangle = 0} \quad . \tag{92}$$

In view of Eqs. (85) and (30) one may determine that the coupling resistance is related to the corresponding loss factor as

$$R_{\alpha j, \beta j} = \eta_{\alpha j, \beta j} \, m_{\alpha j} \, \omega_o \quad . \tag{93}$$

Since this expression is seen to involve the modal mass $m_{\alpha j}$, one observes that in general $R_{\alpha j, \beta k} \neq R_{\beta k, \alpha j}$, although the corresponding two loss factors may be equal.

The loss resistance R_α applicable to a set α of modes should describe the total power loss according to an equation like Eq. (90). The choice of what one defines as the mean square velocity is somewhat arbitrary; it turns out that in dealing with a set of modes of an elastic system of mass m_α it is convenient to choose $\langle V_\alpha^2 \rangle = 2 \, T_{T\alpha} / m_\alpha$. Then one obtains

$$R_\alpha = \frac{D_{T\alpha} \, m_\alpha}{2 \, T_{T\alpha}} = \eta_\alpha \, m_\alpha \, \omega_o \quad , \tag{94}$$

in view of Eq. (84). The above expression is analogous to Eqs. (91) and (93), except that m_α here represents a system mass, rather than a modal mass.[9]

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[9] With the mode shape normalization indicated in Eq. (7), the modal mass is the same as the system mass. But other normalizations may be used legitimately, and with those this equality would not hold.

By analogy to Eq. (94) one may define a coupling resistance $R_{\alpha\beta}$ to account for the loss of power from a set of modes α to a set of modes β as

$$R_{\alpha\beta} \equiv \frac{P_{\alpha\beta}\, m_\alpha}{2\, T_{T\alpha}} \Bigg|_{T_\beta=0} \tag{95}$$

and determine by use of Eqs. (87) and (88) that

$$R_{\alpha\beta} = \eta_{\alpha\beta}\, m_\alpha\, \omega_o = \frac{\phi_{\alpha\beta}\, m_\alpha}{2\, N_\alpha} \quad . \tag{96}$$

Again, $R_{\alpha\beta}$ and $R_{\beta\alpha}$ are generally different, even if $\phi_{\alpha\beta}=\phi_{\beta\alpha}$.

Modal Densities

One usually studies the interactions of elastic systems on a band-by-band basis, by determining the power flow from modes of one system which have their resonances in a given frequency band to a set of modes of another system which have their resonances in the same band. One may in general choose any convenient bandwidth, and one may often wish to use different bandwidths for different calculations. The number of modes which fall within the various sets depends on the bandwidths one chooses, so that it is useful to introduce the concept of modal density, in terms of which the dependence of some of the previous results on bandwidth can be stated more explicitly.

In addition, it turns out that modal densities for many systems can be estimated readily; thus, use of modal densities increases the practical utility of the analysis method discussed in this report.

The modal density of an elastic system is defined as the average number of modes per unit frequency interval. If a system exhibits N_α modes whose resonances fall within a frequency interval $\Delta\omega$, then the modal density of the system at the center frequency ω of the interval is defined as

$$n_\alpha(\omega) = N_\alpha/\Delta\omega \quad . \tag{97}$$

For most purposes it suffices to consider $n_\alpha(\omega)$ as given by Eq. (97) to be a "smooth" function of frequency. Such smoothness may be attained by restricting the applicability of Eq. (97) to cases where the frequency interval of interest encompasses several modes.[10]

The modal densities of systems may be determined from their "frequency equations", i.e., from the equations which give the system resonance frequencies as a function of the system parameters. For simple systems these calculations may be carried out without great difficulty, as illustrated in Ref. 5.

Table I lists the modal densities of some uniform elastic systems, as obtained from Ref. 5 and from calculations based on Ref. 2. The expressions listed in the table apply strictly only for frequencies which are considerably above the system fundamental (by perhaps at least two octaves), where boundary conditions have no important effects; however, these expressions generally also provide reasonable estimates for the modal densities at frequencies only slightly above the fundamental.

The modal density of a composite system is approximately equal to the sum of the modal densities of the component systems. Thus, for example, the modal density of a plate with attached beams is roughly equal to the modal density of the plate by itself, plus the modal densities of all of the beams by themselves. This additive property of modal densities has not been validated analytically, but appears plausible on the basis of some limited experimental evidence and the following reasoning (Ref. 17).

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[10] Otherwise, small changes in ω might result in large stepwise changes in n_α. For example, if $\Delta\omega$ is small, it might encompass one or more resonances for one center frequency, but none for another frequency near the previous one. The assumption of smoothness generally introduces no important errors in estimates of average responses in broad frequency bands. (The errors due to this assumption usually are much smaller than those introduced by other approximations and estimates one usually must make in order to apply the statistical energy approach to practical problems.) However, one may need to concern oneself with the fluctuations in n_α in dealing with more advanced problems concerning the deviations of response from the broad-band average.

TABLE I

MODAL DENSITIES OF SOME UNIFORM SYSTEMS*

System	Motion	Modal Density, $n(\omega)$	Auxiliary Expressions
String	Lateral	$L/\pi c_s$	$c_s = \sqrt{T/\rho A}$
Shaft, Beam	Torsion	$L/\pi c_T$	$c_T = \sqrt{Gk/\rho J}$
Shaft, Beam	Longitudinal	$L/\pi c_\ell$	$c_\ell = \sqrt{E/\rho}$
Beam	Flexure	$\frac{L}{2\pi}\left(\omega\, \kappa_b c_\ell\right)^{-1/2}$	$\kappa_b c_\ell = \sqrt{EI/\rho A}$
Membrane	Lateral	$A_s\, \omega/2\pi c_m^2$	$c_m = \sqrt{S/\rho h}$
Plate	Flexure	$A_s/4\pi\, \kappa_p c_\ell$	$\kappa_p c_\ell = \sqrt{D/\rho h} = \sqrt{Eh^2/12\rho(1-\nu^2)}$
Room, (Acoustic Volume)	Sound (Compression)	$V_o\omega^2/2\pi^2\, c_a^3$	
Cylindrical Shells (Ref. 15)	Flexure	$\approx n_p$ for $\omega/\omega_r > 1$ $\approx n_p\left(\frac{\omega}{\omega_r}\right)^{2/3}$ for $\omega/\omega_r < 1$	$\omega_r = c_\ell/a$ $n_p = A_s/4\pi\, \kappa_p c_\ell$
Doubly Curved Shells	Flexure	Expressions are complex, given in Ref. 16.	

*See next page for definitions of symbols.

Symbol Definitions for Table I

A	cross-section area
A_s	surface area
c_a	acoustic wave velocity
c_ℓ	longitudinal wave velocity
c_m	membrane wave velocity
c_s	string wave velocity
c_T	torsional wave velocity
D	plate rigidity
E	Young's modulus
G	shear modulus
h	thickness
I	centroidal moment of inertia of A
J	polar moment of inertia of A
K	torsional constant of A
L	length
S	membrane tension force/unit edge length
T	string tension force
V_o	volume
κ_b	radius of gyration of A
κ_p	radius of gyration of plate cross section
ν	Poisson's ratio
ω	frequency (radians/time)
ρ	material density

The number of modes in a lumped parameter system (e.g., one consisting of ideal springs, masses, and dashpots) is equal to the number of degrees of freedom. If one interconnects two such systems, then the total number of degrees of freedom of the composite system is equal (or very nearly equal, depending on the manner of interconnection) to the sum of the number of degrees of freedom of the individual systems. Thus, the number of modes of the combined system is equal to the sum of the numbers of modes of the two constitutent systems.

However, some additional considerations are required, since the modal resonances of the combined system generally occur at different frequencies than the resonances of the individual systems. If one interconnects two single-degree-of-freedom systems having different resonant frequencies, then one obtains a two-degree-of-freedom system which has one resonance below the lower of the two individual system resonances, and one which is above the higher of the individual resonances. In other words, the interconnection in effect shifts the resonances both upward and downward. If two multimodal systems are interconnected, it is likely that on the average as many resonances are shifted into a fixed frequency band as are shifted out of it, so that the total number of modes in the band is unchanged by the interconnection process.

SOME SPECIAL RESULTS FOR TWO COUPLED MODE SETS

For two coupled mode sets, where one of the sets is not directly excited, use of the statistical energy method is particularly fruitful and can lead to valuable qualitative insights, as well as to quantitative results. The present section summarizes general results applicable to the aforementioned case, points out some conclusions one may draw from them, and illustrates application of these results to some special cases.

Vibrations of Indirectly Excited Mode Sets

Consider two coupled sets of modes, as represented schematically in Fig. 4, and assume that $A_\beta=0$, - that is, that no energy is supplied to mode set β except via its coupling to mode set α. In the steady state the total power $D_{T\beta}$ dissipated by set β must be equal to the power $P_{\alpha\beta}$ supplied to it from set α. If the modes of each set are sufficiently "similar", in the sense indicated in the previous chapter, so that one may validly use Eq. (61) with $\phi_{\alpha\beta}=\phi_{\beta\alpha}$ to describe $P_{\alpha\beta}$, then the energy balance for set β requires that

$$\phi_{\alpha\beta}(T_\alpha - T_\beta) = 2\,\omega_o \eta_\beta N_\beta T_\beta \quad , \tag{98}$$

in view of Eq. (84). If one solves this expression for T_β one obtains a relation between the average modal kinetic energies of the two sets which may be expressed as

$$\boxed{\frac{T_\beta}{T_\alpha} = \frac{(\phi_{\alpha\beta}/2\omega_o)}{(\phi_{\alpha\beta}/2\omega_o) + \eta_\beta N_\beta} = \frac{\eta_{\beta\alpha}}{\eta_{\beta\alpha} + \eta_\beta}} \tag{99}$$

where Eq. (89) has been used to replace $\phi_{\alpha\beta}$ by the coupling loss factor[11] $\eta_{\beta\alpha}$.

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[11] It should be noted that Eq. (99) involves $\eta_{\beta\alpha}$, the loss factor which pertains to energy flow from β to α, whereas the actual energy flow must be in the other direction if set β is not directly excited.

As indicated in the discussion pertaining to Eqs. (17) and (18), the space- and time-average mean square velocity $\langle\bar{v}_\alpha^2\rangle$ associated with N_α modes of an elastic system of mass m_α is related to the total kinetic energy $T_{T\alpha}$ of these modes as

$$T_{T\alpha} = N_\alpha T_\alpha = \frac{1}{2} m_\alpha \langle\bar{v}_\alpha^2\rangle \quad . \tag{100}$$

If one uses this expression and an analogous one for set β to eliminate the average modal energies T_α and T_β from Eq. (99), one obtains the result

$$\boxed{\frac{\langle\bar{v}_\beta^2\rangle}{\langle\bar{v}_\alpha^2\rangle} = \frac{N_\beta}{N_\alpha} \cdot \frac{m_\alpha}{m_\beta} \cdot \frac{\eta_{\beta\alpha}}{\eta_{\beta\alpha}+\eta_\beta} = \frac{n_\beta(\omega)}{n_\alpha(\omega)} \cdot \frac{m_\alpha}{m_\beta} \cdot \frac{\eta_{\beta\alpha}}{\eta_{\beta\alpha}+\eta_\beta}} \; . \tag{101}$$

The last form of this expression is obtained by use of Eq. (97) and is based on the assumption that the N_α and N_β modes correspond to the same frequency band $\Delta\omega$.

Equation (101) is a surprisingly simple and useful result, which permits one to calculate the mean square velocity of the indirectly excited elastic system from that of the directly excited one if one knows the masses of the two systems, their modal densities (e.g., from Table I) in the various frequency bands of interest, and the coupling and dissipation loss factors. A further simplification occurs for those cases where $\eta_\beta \ll \eta_{\beta\alpha}$, since then Eq. (101) reduces to a relation which does not involve the loss factors. Applications of Eq. (101) to a beam coupled to a plate and to two coupled plates, and related experimental results showing generally good agreement with theoretical predictions, are presented in Ref. 18.

The grouping of the loss factor terms appearing in Eqs. (99) and (101) permits one to make some observations concerning the effectiveness of adding damping to indirectly excited systems. If the damping of the β system is small initially, that is, if $\eta_\beta \ll \eta_{\beta\alpha}$, then added damping will reduce the system vibrations $\langle\bar{v}_\beta^2\rangle$ only if this added damping (or increase in η_β) is significant as compared to $\eta_{\beta\alpha}$. This observation may explain why the addition of some damping to a structure (system β) excited by sound in an acoustic space (system α) may have little effect on some components of the structural vibrations.

On the other hand, Eq. (101) indicates that the mean square velocity of the indirectly excited system varies approximately inversely with the loss factor η_β in those cases where $\eta_\beta >> \eta_{\beta\alpha}$. Thus, added damping is effective in reducing vibrations of indirectly excited systems, provided that the resulting dissipation loss factor exceeds the coupling loss factor.

From Eq. (99) or (98) one may conclude that $T_\beta \approx T_\alpha$ if $\eta_{\beta\alpha} >> \eta_\beta$. That is, if the power dissipation of set β is negligible, then so is the power flow from set α to set β, and the average kinetic energy T_β of the indirectly excited set will approach that of the directly excited set. Thus, if $\eta_{\beta\alpha} >> \eta_\beta$, then the two mode sets will approach "modal energy equilibrium" or "modal energy equipartition".

Effect of Coupling on Vibrations of Directly Excited Mode Sets

The present discussion, like that of the previous section, deals with two coupled sets of modes, whose interaction may be visualized by means of the lower portion of Fig. 3. Here, as in the previous section, no energy is assumed to be supplied to set β from outside the system; i.e. $A_\beta = 0$. However, unlike the previous section, which focused on the indirectly excited set β, the present section is concerned with the directly excited set α.

Conservation of energy requires that the time-average power A_α supplied to set α in the steady state be equal to the total power dissipated by both mode sets. As previously pointed out, the power dissipated by set β must be equal to the power supplied to it from set α. Thus, one may write

$$A_\alpha = 2\omega_o \eta_\alpha N_\alpha T_\alpha + \phi_{\alpha\beta}(T_\alpha - T_\beta) \tag{102}$$

$$= 2\omega_o T_\alpha \left[\eta_\alpha N_\alpha + \frac{\phi_{\alpha\beta}}{2\omega_o} \left(1 - \frac{T_\beta}{T_\alpha} \right) \right] .$$

If one eliminates T_β by substituting Eq. (99), and if one introduces the coupling loss factors by use of Eq. (89), then one may find that the foregoing results in

$$\frac{A_\alpha}{2\omega_o T_\alpha N_\alpha} \equiv \eta_{\alpha_{app}} = \eta_\alpha + \frac{\eta_{\alpha\beta}\,\eta_\beta}{\eta_{\beta\alpha} + \eta_\beta} \qquad (103)$$

The "apparent loss factor" $\eta_{\alpha_{app}}$ introduced in the foregoing equation corresponds to the value of the dissipation loss factor which one would ascribe to set α (on the basis of measurements performed on set α) if one were not aware that this set is coupled to set β. In order to determine the loss factor η_α of set α in absence of coupling, one would turn to Eq. (84) and determine the values of the various parameters N_α, T_α, $D_{T\alpha}$ that appear there. Since the directly supplied average power A_α must be equal to the total dissipated $D_{T\alpha}$ (in absence of coupling to other mode sets), one may replace $D_{T\alpha}$ by A_α; then the ratio appearing in Eq. (84) becomes precisely that which occurs on the left-hand side of Eq. (103).

It is evident from Eq. (103) that the apparent loss factor $\eta_{\alpha_{app}}$ is never smaller than the actual dissipation loss factor η_α. Thus, if one is unaware that a system whose loss factor one is measuring is coupled to another system, then one always obtains a loss factor value which is too large. The error is insignificant, however, if the coupling is poor ($\eta_{\alpha\beta}, \eta_{\beta\alpha} \approx 0$) and/or if the coupled set is relatively lossless ($\eta_\beta << \eta_\alpha$). Effects of energy conduction on loss-factor measurements are discussed in some more detail in Refs. 19 and 20.

From Eq. (103) one may also deduce that the response $\langle \overline{V}_\alpha^2 \rangle = 2N_\alpha T_\alpha / m_\alpha$ to a given excitation A_α is not controlled by the actual dissipation loss factor η_α (as in the uncoupled case), but rather by the apparent loss factor $\eta_{\alpha_{app}}$. Thus, the coupled indirectly excited system β increases the effective damping of the directly excited system α; - a conclusion which is intuitively "obvious", since system β here serves to extract energy from system α.

Interaction of Sound and Structures

Some of the previous results of this section are particularly useful for estimating the sound produced in an acoustic volume due to the vibration of a structure, or for predicting the structural vibrations induced by a diffuse sound field.

Consider the case where a structure S is directly excited by random forces acting on it, and where these structural vibrations produce noise in a room R in which the structure is located. Here the structural modes in a frequency band of interest correspond to a directly excited set α, the modes of the acoustic space R correspond to the indirectly excited set β.

The total energy E_R in a diffuse acoustic field within fluid volume V_o in a frequency band $\Delta\omega$ is related to the mean square acoustic pressure $\langle\bar{p}^2\rangle$ measured in the same frequency band as (p. 95 of Ref. 5)

$$E_R = \langle\bar{p}^2\rangle \, V_o/\rho_a c_a^2 \tag{104}$$

where ρ_a denotes the density and c_a the sound velocity of the fluid. This total energy is on the average half potential and half kinetic, hence the average total kinetic energy T_{TR} is one half of the total. The average modal kinetic energy T_R may be obtained by dividing T_{TR} by $n_R \cdot \Delta\omega$, the number of modes in the interval. With the modal density expression given in Table I (p. 48) for acoustic volumes, one thus finds the average kinetic energy of an acoustic mode of a room to be given by

$$T_R = \frac{E_R}{2\, n_R\, \Delta\omega} = \frac{\pi^2}{\rho_a}\frac{c_a}{\omega^2}\frac{\langle\bar{p}^2\rangle}{\Delta\omega} \quad . \tag{105}$$

If one applies Eq. (100) to the structure under consideration and replaces the number of modes N_S by $n_s \cdot \Delta\omega$ according to Eq. (97), then one finds that the average kinetic energy of a structural mode is given by

$$T_S = \frac{m_s\langle\bar{v}_s^2\rangle}{2\, n_s \cdot \Delta\omega} = \frac{m_s\langle\bar{a}_s^2\rangle}{2\,\omega^2\, n_s\, \Delta\omega} \tag{106}$$

where $\langle\bar{v}_s^2\rangle$ represents the mean square velocity and $\langle\bar{a}^2\rangle = \omega^2\langle\bar{v}^2\rangle$ the mean square acceleration of the structure measured in the frequency band $\Delta\omega$ with center frequency ω.

Introduction of Eqs. (105) and (106) into Eq. (99), with R taking the place of the indirectly excited system β and S that of the directly excited system α, leads to the following relation between the mean square values of the acoustic pressure and the structural acceleration:

$$\frac{\overline{\langle p^2 \rangle}}{\overline{\langle a_s^2 \rangle}} = \frac{\rho_a \, m_s}{2\pi^2 c_a n_s} \frac{\eta_{RS}}{\eta_{RS}+\eta_R} \tag{107}$$

In most cases the room-to-structure coupling loss factor η_{RS} may be expected to be much smaller than the dissipation loss factor of the room. With this inequality, and with

$$\eta_{RS} = \frac{\eta_{SR} \, n_s}{n_R} = \frac{n_s \, R_{SR}}{n_R \, m_s \, \omega} \tag{108}$$

obtained from Eqs. (89), (97), (96), one finds that one may rewrite Eq. (107) as

$$\frac{\overline{\langle p^2 \rangle}}{\overline{\langle a_s^2 \rangle}} = \frac{\rho_a \, R_{SR}}{2\pi^2 c_a \, \omega \, n_R \eta_R} = \frac{R_{SR}}{\eta_R} \cdot \frac{\rho_a \, c_a^2}{V_o \omega^3} \quad , \tag{109}$$

where the last form is obtained by substituting for n_R the appropriate expression from Table I. The structure-to-room coupling resistance R_{SR} corresponds to the "radiation resistance" which acts on the structure; i.e. to the resistance which governs the loss of energy by the structure due to acoustic radiation from it.

One may also consider a situation in which an acoustic volume R is directly excited by a noise source, and where the resulting acoustic pressures cause an (indirectly excited) structure to vibrate. Here the volume excites the structure; i.e. the roles of the two systems here are interchanged from the previously discussed case. Equations (104)-(106) still apply, but now the structure S corresponds to the indirectly excited set β and the room R to the indirectly excited set α. Here, use of Eq. (99) yields a result which one may write

$$\frac{\overline{\langle a_s^2 \rangle}}{\overline{\langle p^2 \rangle}} = 2\pi^2 \frac{c_a}{\rho_a} \frac{n_s}{m_s} \mu \tag{110}$$

where

$$\mu \equiv \frac{\eta_{SR}}{\eta_{SR}+\eta_S} = \frac{R_{SR}}{R_{SR}+R_S} \tag{111}$$

and the last form of Eq. (111) follows from Eq. (96).

The results given in Eqs. (109)-(111) first appeared in Ref. 10. This reference also describes some experimental results, which are in good agreement with theoretical predictions.

A word of caution is required, however, concerning the applicability of the results of this section. These results are derived from Eqs. (98) and (99), which in turn are based on the assumption that a simple power flow relation holds for the two mode sets considered. Thus, Eqs. (109)-(111) should only be applied to cases where all structural modes in the frequency band of interest are uniformly coupled to all room modes in the band. If differently excited or differently coupled modes occur in the band, one must divide them into groups of like modes, and apply the foregoing results to each group separately, of course using properly modified modal density expressions, as needed. Differences in sound-structure coupling and calculations taking these into account are discussed in Ref. 5.

CONCLUDING REMARKS

It is hoped that this report will introduce the reader to the basic concepts involved in the statistical energy analysis of vibrating systems, so that he may put this approach to use with some confidence. It was endeavored to describe the most important aspects of this analysis approach in considerable detail, in order to give the reader an understanding of its range of validity. Of course, not all aspects of available results pertinent to this approach could be covered within the time and space allotted to this report; the reader interested in particular topics may do well to consult some of the references given here, as well as some of the background information cited in these references.

Although the results presented in this report are limited to linear coupling, a study of nonlinearly coupled modes has been published recently (Ref. 21). This study shows that the net energy flow between nonlinearly coupled modes also occurs in the direction of the energy gradient, and that no energy flows if the modal energies are equal.

The present report has dealt essentially only with two coupled modes, or with two coupled mode sets, but not with more sets or modes in cascade. For loose coupling, extension of the two-set results to multiset-systems can readily be accomplished. Such extensions and applications of them appear in Refs. 22-24, and a complete electric-circuit analogy for energy flow in multiset-systems is developed in Ref. 25. As yet unpublished calculations indicate that the power flow between any two adjacent modes of three linearly coupled modes in cascade is proportional to the modal energy difference, even if the modes are well coupled; however, here the factors of proportionality generally involve properties of all three modes and thus differ from the two-coupled-mode case.

Calculations on the basis of the statistical energy approach of variances and confidence limits, in addition to the commonly computed averages of the dynamic variables, are presented in Ref. 18. Illustrations of applications of the statistical energy approach and discussions of the problems which have been studied by means of it appear in Refs. 17-19, 22-28.

At present the usefulness of the statistical energy approach in predicting vibration responses under practical circumstances is limited by the amount of information that is available concerning coupling coefficients and loss factors. Approximate determination of sound-to-structure coupling

coefficients (or resistances) often is not very difficult, as indicated in Ref. 5, but calculations of structure-to-structure coupling coefficients may involve considerable effort (e.g., see Ref. 18). A reasonable amount of information on the dissipative loss factors of materials and simple structures is available in the technical literature (e.g., Ref. 13), but the loss factors of built-up structures can still not be estimated with much confidence - although some progress is being made (Ref. 29).

The statistical energy approach is a uniquely simple and powerful tool for providing one with a qualitative understanding of the most important aspects of the vibrations of complex systems. When combined with other, analytically or experimentally derived, information concerning the energy transport and dissipation parameters involved in a given case, this approach also permits one to arrive readily at quantitative answers to complex vibration problems.

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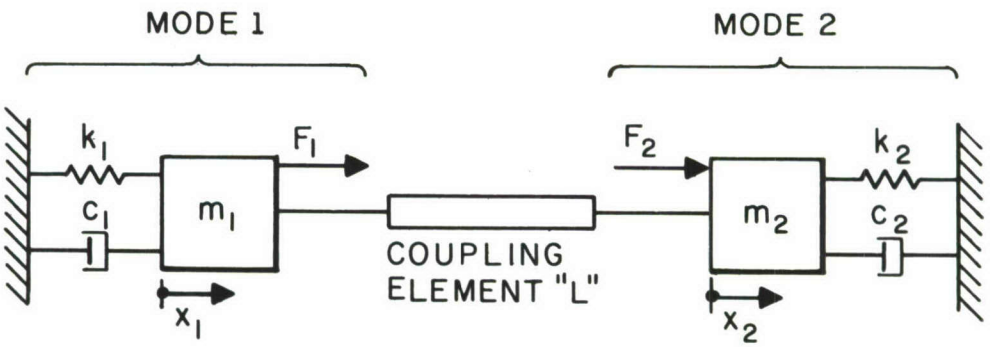


FIG. 1 TWO COUPLED MODES

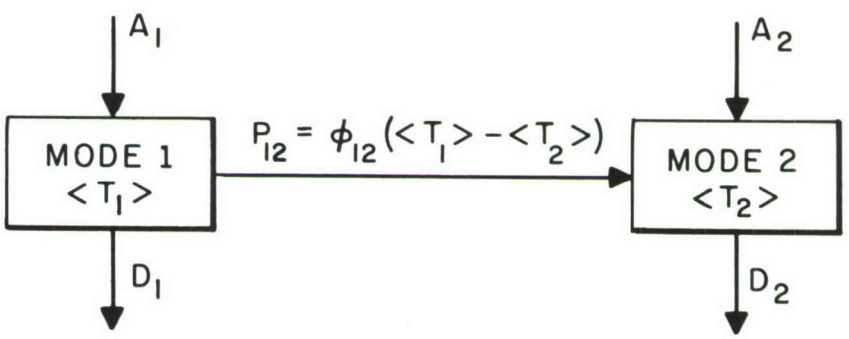


FIG. 2 POWER FLOW DIAGRAM FOR TWO COUPLED MODES

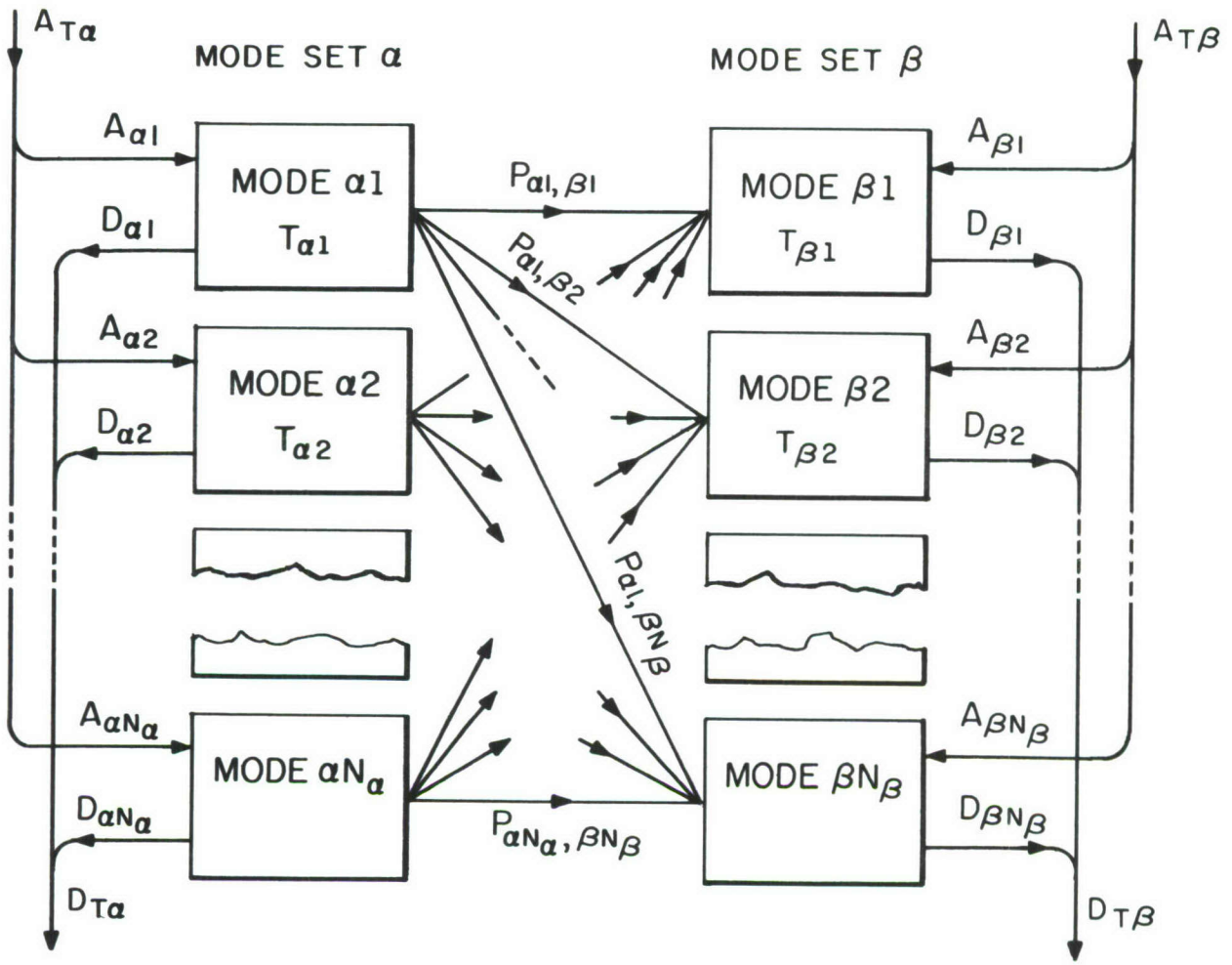
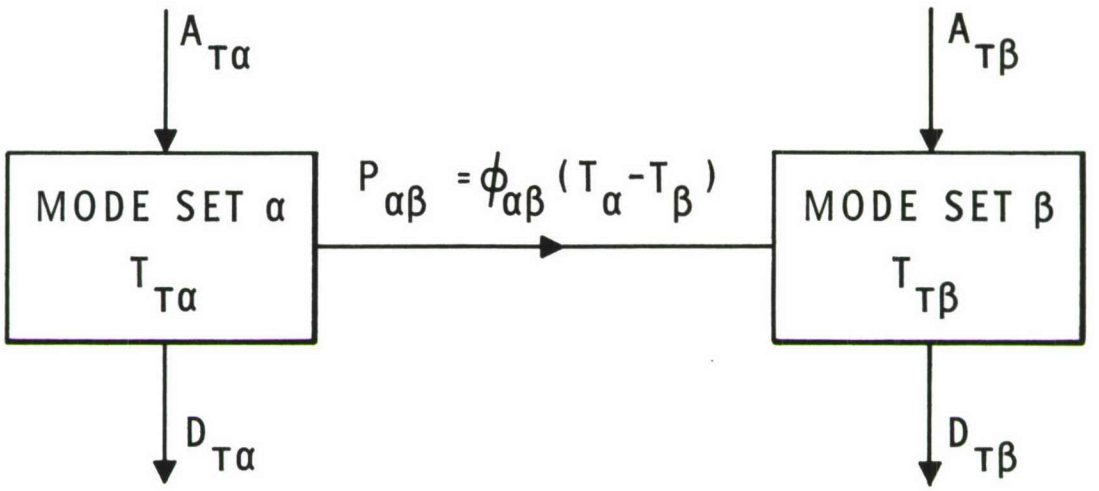


FIG. 3 TWO COUPLED SETS OF MODES



	Ref. Eqs. (74)
$A_{T\alpha} = N_\alpha A_\alpha \approx \pi N_\alpha S_\alpha / m_\alpha$	
$D_{T\alpha} = N_\alpha D_\alpha = 2\omega_o \eta_\alpha T_{T\alpha}$	(83),(84)
$T_{T\alpha} = N_\alpha T_\alpha = \langle \bar{V}_\alpha^2 \rangle m_\alpha/2 = N_\alpha \langle V_\alpha^2 \rangle m_\alpha/2$	(17),(18)
$P_{\alpha\beta} = \phi_{\alpha\beta}(T_\alpha - T_\beta)$	(61),(68),(69),(79)
$\phi_{\alpha\beta} = N_\alpha N_\beta \phi_o = 2\omega_o N_\alpha \eta_{\alpha\beta} = 2\omega_o N_\beta \eta_{\beta\alpha}$	(68),(88)

See next page for definitions of symbols.
See pp. 39, 40 for summary of restrictions under which the above relations apply.

FIG. 4 POWER FLOW BETWEEN TWO SETS OF MODES

SYMBOL DEFINITIONS FOR FIG. 4

The following abbreviations are used in this list of symbols:

t.a.	time-average	tot.	total
s.a.	set-average	m.s.α.	mode(s) of set α

Symbols	Definitions
$A_{T\alpha}$	t.a. tot. power supplied to all m.s.α.
$D_{T\alpha}$	t.a. tot. power dissipated by all m.s.α.
$T_{T\alpha}$	t.a. tot. kinetic energy of all m.s.α.
A_α	s.a. of t.a. power supplied to single m.s.α.
D_α	s.a. of t.a. power dissipated by single m.s.α.
T_α	s.a. of t.a. kinetic energy of single m.s.α.
N_α	number of modes in set α
m_α	tot. mass of elastic system containing the m.s.α.
$\langle \bar{V}_\alpha^2 \rangle$	t.a. and space-average mean-square velocity (of elastic system containing set α) due to all m.s.α.
$\langle V_\alpha^2 \rangle$	t.a. and s.a. mean square velocity of single m.s.α.
S_α	s.a. spectral density of forces acting on m.s.α.
η_α	dissipation loss factor for set α
$\eta_{\alpha\beta}$	loss factor pertaining to power flow from set α to set β
ω_o	center frequency of band containing all modal resonances of both sets
$\phi_{\alpha\beta}$	set-to-set power flow coefficient
ϕ_o	mode-to-mode power flow coefficient (averaged over both sets)

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13. ABSTRACT

The concepts and relations which form the theoretical foundation of the statistical energy approach to vibration analysis are delineated. The utility of this approach for dealing with complex systems is discussed, and its range of applicability is indicated. Examples are presented which illustrate how the mode-to-mode power flow relations developed in the report permit one to obtain approximate solutions to complex problems simply, on the basis of energy conservation considerations.

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	ROLE	WT	ROLE	WT	ROLE	WT
Vibrations Vibratory Systems Sound Transmission						

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